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Spinor Relativity

Saskia Kind

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Abstract Spinor relativity is a unified field theory, which derives gravitational and electromagnetic fields as well as a spinor field from the geometry of an eight-dimensional complex and 'chiral' manifold. The structure of the theory is analogous to that of general relativity: it is based on a metric with invariance group $\mathbf{GL}(\mathbb{C}^2)$, which combines the Lorentz group with electromagnetic $\mathbf{U}(1)$, and the dynamics is determined by an action, which is an integral of a curvature scalar and does not contain coupling constants. The theory is related to physics on spacetime by the assumption of a symmetry-breaking ground state such that a four-dimensional submanifold with classical properties arises. In the vicinity of the ground state, the scale of which is of Planck order, the equation system of spinor relativity reduces to the usual Einstein and Maxwell equations describing gravitational and electromagnetic fields coupled to a Dirac spinor field, which satisfies a non-linear equation; an additional equation relates the electromagnetic field to the polarization of the ground state condensate.

Keywords General relativity · Spinors · Unification · Symmetry breaking · Extra dimensions

1 Introduction and Summary of Results

In covariant approaches to quantum gravity the spacetime metric g is usually decomposed into a background metric, which is considered as a classical field, and a deviation h from this background, which is quantized. In the simplest case the Minkowski metric is taken as background:

$$g_{\mu\nu} = \eta_{\mu\nu} + \lambda_P h_{\mu\nu} \tag{1.1}$$

One may consider such a theory as arising from a more fundamental theory with quantized g by the development of a vacuum expectation value:

$$\langle g_{\mu\nu} \rangle_0 = \eta_{\mu\nu} \tag{1.2}$$

S. Kind (✉)
Elsenheimerstraße 32, 80687 Munich, Germany
e-mail: kind@mppmu.mpg.de

h and the other fields of the theory then describe excitations above this vacuum. In quantum gravity the quantized metric g is defined on a real four-dimensional manifold.

The starting point of spinor relativity is an eight-dimensional 'chiral' manifold, as described below. At the classical level the theory is analogous to general relativity, i.e. it is based on a metric as fundamental field, from which a connection is derived, and the dynamics of the metric is determined by an action functional with a curvature scalar in its integrand. However, at the quantum level the metric is supposed to be far from classical behaviour and the eight-dimensional manifold may not be considered as a classical manifold. The possibility is investigated that the theory develops a symmetry-breaking ground state such that a preferred four-dimensional submanifold arises, which attains classical properties and is identified with spacetime. The reason that such an approach might be useful is that it opens new possibilities for unification. In spinor relativity the dynamics is determined by an action functional consisting of a single term. The usual equations with coupling constants arise as an effective theory describing low energy excitations above the ground state, the scale of which is of Planck order. In particular, the kinetic term $\sim F^2$ in the Lagrangian of the electromagnetic field is considered as not being fundamental.

In order to see how an alternative derivation of the Maxwell equations might be possible, it is useful to investigate a $3+1$ decomposition of linearized gravity on Minkowski spacetime. The Riemann tensor decomposes into four symmetric spatial tensors and four vectors:

$$\begin{aligned} R_{m0n0} &= \mathbf{E}_{mn} + \epsilon_{mnk} B_k \\ \frac{1}{2} R_{m0rs}\epsilon_{rsn} &= \mathbf{B}_{mn} - \epsilon_{mnk} E_k \\ \frac{1}{2}\epsilon_{mrs} R_{rsn0} &= \mathbf{H}_{mn} - \epsilon_{mnk} D_k \\ -\frac{1}{4}\epsilon_{mrs} R_{rspq}\epsilon_{pqn} &= \mathbf{D}_{mn} + \epsilon_{mnk} H_k \end{aligned} \tag{1.3}$$

This decomposition is valid for the Riemann tensor of an arbitrary metric connection, which need not be torsion-free. The requirement of vanishing torsion in general relativity yields additional symmetries of the Riemann tensor, which enforce a vanishing of in particular 'magnetic' components:

$$\mathbf{H} = \mathbf{B}, \qquad H = B = 0, \qquad D + E = 0, \qquad \mathrm{Tr}\mathbf{B} = 0 \tag{1.4}$$

The Einstein equation $G_{\alpha\beta} = 8\pi\lambda_P^2 T_{\alpha\beta}$ then relates the 'electric' components of the Riemann tensor with matter fields

$$\mathbf{D} - \mathbf{E} + \mathrm{Tr}\mathbf{E} = \Theta, \qquad D - E = M, \qquad \mathrm{Tr}\mathbf{D} = -\mu \tag{1.5}$$

where μ is the energy density, M the energy flux and Θ the stress tensor of matter, and the gravitational coupling constant has been absorbed into the definition of these fields for simplicity. The linearized Bianchi identity $R_{\alpha\beta[\gamma\delta,\epsilon]} = 0$ for a general Riemann tensor (1.3) decomposes as follows

$$\begin{aligned} &\mathrm{Div}\mathbf{B} - \mathrm{rot}E = 0 && \mathrm{Div}\mathbf{D} + \mathrm{rot}H = 0 \\ &\dot{\mathbf{B}} + \mathrm{Rot}\mathbf{E} + \widehat{\mathrm{T}}\mathrm{en}B = 0 && \dot{\mathbf{D}} - \mathrm{Rot}\mathbf{H} + \widehat{\mathrm{T}}\mathrm{en}D = 0 \\ &2\dot{E} - \mathrm{rot}B + \mathrm{Div}\widehat{\mathbf{E}} = 0 && 2\dot{H} + \mathrm{rot}D + \mathrm{Div}\widehat{\mathbf{H}} = 0 \end{aligned} \tag{1.6}$$

where the tensor differential operators generalize the usual rotation and divergence of a spatial vector, and a 'hat' on a spatial tensor denotes subtraction of its trace.[1]

The linearized Bianchi identity thus decomposes into two independent equation systems, the first of which involves fields of type E and B only, while the second involves fields of type D and H. Each of these equation systems resembles the homogenous Maxwell equations. Equations (1.4) and (1.5) may now be used to eliminate some of the components of the Riemann tensor in favour of matter fields. This yields besides energy and momentum conservation ($\dot{\mu} + \mathrm{div}M = 0$ and $\dot{M} + \mathrm{Div}\Theta = 0$) equations for the trace-free parts of the fields $\mathbf{E}$ and $\mathbf{B}$, which are denoted by a subscript zero

$$\begin{aligned} &\mathrm{Div}\mathbf{B}_0 = J_m \qquad &&\mathrm{Div}\mathbf{E}_0 = J_e \\ &\dot{\mathbf{B}}_0 + \mathrm{Rot}\mathbf{E}_0 = 0 \qquad &&\dot{\mathbf{E}}_0 - \mathrm{Rot}\mathbf{B}_0 = -\mathbf{J} \end{aligned} \tag{1.7}$$

where the 'gravitational currents' on the right hand sides of these equations are defined by:

$$J_e = -\mathrm{Div}\Theta_0 + \frac{1}{3}\nabla\mu, \qquad J_m = -\frac{1}{2}\mathrm{rot}M, \qquad \mathbf{J} = \dot{\Theta}_0 - \frac{1}{3}\dot{\mu} + \frac{1}{2}\widehat{\mathrm{Ten}}M \tag{1.8}$$

Equation (1.7) resembles the complete, i.e. homogenous and inhomogenous Maxwell equations, this time for one set of fields only.

These investigations suggest that at a fundamental level there is in addition to the electromagnetic field F an 'axial' field $\widetilde{F}$ with components $\widetilde{F}_{k0} = H_k$ and $-\frac{1}{2}\epsilon_{kmn}\widetilde{F}_{mn} = D_k$. The Bianchi identities of these fields yield two sets of homogeneous Maxwell equations in analogy to (1.6):

$$\begin{aligned} &\mathrm{div}B = 0 \qquad &&\mathrm{div}D = 0 \\ &\dot{B} + \mathrm{rot}E = 0 \qquad &&\dot{D} - \mathrm{rot}H = 0 \end{aligned} \tag{1.9}$$

The electromagnetic field tensors are related to matter by an 'electromagnetic Einstein equation'

$$F_{\alpha\beta} + \widetilde{F}^*_{\alpha\beta} = P_{\alpha\beta} \tag{1.10}$$

where P is the polarization tensor. This equation relates the fields $D - E$ and $B - H$ to electric polarization and magnetization respectively and may be used to eliminate D and H from the Bianchi identities (1.9) yielding the complete Maxwell equations for E and B with current density given by the divergence of the polarization tensor.

In spinor relativity an axial field is present in form of the gauge field of a $\mathbf{D}(1)$-symmetry of the theory, and the derivation of the Maxwell equations proceeds along the lines indicated above. However, although this formally resembles the macroscopic description of polarizable matter in terms of four field vectors, there is an essential difference concerning the significance of the polarization tensor. Polarization is usually defined as the reaction of a medium to an external electromagnetic field. This is also the case for vacuum polarization, where the bare electromagnetic field polarizes the virtual fermion-antifermion pairs of the vacuum. In spinor relativity on the other hand, there is no bare electromagnetic field; rather,

[1] $(\mathrm{Div}\mathbf{M})_k = \mathbf{M}_{k(m,m)}$, $(\mathrm{Rot}\mathbf{M})_{mn} = \epsilon_{rs(m}\mathbf{M}_{n)s,r}$, $(\widehat{\mathrm{Ten}}V)_{mn} = V_{(m,n)} - \delta_{mn}V_{(k,k)}$, $\widehat{\mathbf{M}} = \mathbf{M} - \mathrm{Tr}\mathbf{M}$.

the electromagnetic field is considered as a composite state of a fundamental spinor field Ψ, which determines the polarization tensor by an equation of the form:

$$P_{\alpha\beta} \sim \bar{\Psi}[\gamma_\alpha, \gamma_\beta]\Psi \tag{1.11}$$

Ψ is not the field of a particular fermion of definite mass, but is comparable to the spinor field in Heisenberg's unified field theory [1]. In this theory Heisenberg introduces a spinor-isopinor field, which satisfies a non-linear differential equation. The free field operators, which describe the elementary particles, Heisenberg considers as secondary objects, which are contained asymptotically in functionals of the fundamental spinor field. This point of view of free field operators may be compared with the quantum mechanical description of the hydrogen atom: here the 'free field operators' create and annihilate the stationary states; these operators are functions of the fundamental operators for position and momentum. The Heisenberg field is somewhat similar to the quark field; however, an essential difference is that QCD assumes the existence of independent gauge particles, which are not composites of the quarks.

The field Ψ of spinor relativity is inherent in the structures of the chiral manifold, as described below. Its square $\bar{\Psi}\Psi$ is assumed to develop a ground state expectation value, which consistency requires to be of Planck order $\sim \lambda_P^{-3}$. One may imagine the ground state as a condensate of fermion-antifermion pairs of the field Ψ and interpret (1.11) as describing the polarization of these pairs, which generates the electromagnetic field in a similar way as the spins of the electrons generate the magnetic field of a ferromagnet.

Spinor relativity is defined on a *chiral manifold* M_{ch}. Such a manifold is the product of two diffeomorphic manifolds M_r and M_l, which are called the right and left manifold respectively, equipped with a *chiral structure*, which is analogous to a complex structure. While the latter structure is preserved by holomorphic coordinate transformations, the structure-preserving coordinate transformations on a chiral manifold are those, which leave the right and left manifolds separately invariant. One may imagine a chiral manifold with M_r and M_l forming a 'double layer'; the points of the chiral manifold consist of arbitrary pairs of points p_r and p_l on the individual manifolds. This picture is convenient, since the ground state of the theory is supposed to break the chiral symmetry of independent coordinate transformations on the right and left manifold, leaving unbroken the symmetry under joint coordinate transformations. The ground state thus establishes a preferred correspondence between points on the right and left manifold. These preferred pairs (p_r, p_l) define a submanifold $\mathcal{S}$ of the chiral manifold, which will be identified with spacetime.

General relativity may be considered as a gauge theory of the Lorentz group.[2] There are distinguished bases in the tangent bundle of the spacetime manifold, the orthonormal tetrad bases, with respect to which the metric takes Minkowski form:

$$g_{\mu\nu} e^\mu{}_\alpha e^\nu{}_\beta = \eta_{\alpha\beta} \tag{1.12}$$

These bases are not unique, there is a gauge freedom to perform local Lorentz transformations on the tetrads. The Levi-Civita connection of the metric provides a gauge covariant derivative. Since spinor relativity aims at a unification of gravitation and electromagnetism,

[2] The Lorentz gauge fields are however not independent but determined by the metric. In order to interpret gravity as a gauge theory of the Lorentz or Poincare group, Einstein-Cartan theory seems to be more appropriate [2].

its gauge group should contain both the Lorentz group and electromagnetic $\mathbf{U}(1)$. A preferred group of this type is

$$\mathbf{GL}(\mathbb{C}^2) = \mathbf{D}(1) \times \mathbf{U}(1) \circ \mathbf{SL}(\mathbb{C}^2) \tag{1.13}$$

which already plays an important part in physics: it is isomorphic to the group of Dirac spinor transformations, which leave the Dirac product $\bar{\psi}\psi$ invariant and commute with the chiral projectors.

In order to construct a $\mathbf{GL}(\mathbb{C}^2)$-gauge theory on a chiral manifold, the right and left manifolds are each chosen as complex manifolds of dimension two, i.e. their real dimension is four, as it should be in order to identify the submanifold $\mathcal{S}$ with spacetime. The tangent bundle of the chiral manifold is further equipped with a symmetric inner product, which between right sections $\phi_r \in W(M_r)$ and complex conjugate left sections $\bar{\phi}_l \in \bar{W}(M_l)$ takes the form

$$(\bar{\phi}_l, \phi_r) = \bar{\phi}_l^{\dot{M}} g_{\dot{M}N} \phi_r^N \tag{1.14}$$

where dotted indices refer to the complex conjugate tangent spaces. $g_{\dot{M}N}$, which in general is not hermitian, is called the *Dirac metric*. All fields are functions of pairs of points on M_r and M_l, e.g. the value of $\phi_r \in W(M_r)$ also depends on the point chosen on M_l. The inner product between two right sections or between two left sections is zero, which in particular means that there is no (pseudo-)riemannian or hermitian structure on the individual manifolds M_r and M_l.

In analogy to orthonormal tetrad bases on spacetime, *spinor bases* may be introduced on the chiral manifold, which are defined by the requirement that the Dirac metric take the form of a unit matrix:

$$g_{\dot{M}N} \bar{E}_l^{\dot{M}}{}_{\dot{A}} E_r^N{}_B = \delta_{\dot{A}B} \tag{1.15}$$

The transformation matrices E_r and E_l, which transform from coordinate bases to spinor bases in the right and left tangent bundle respectively, are called *dyad fields*, in analogy to the tetrad fields of general relativity. The gauge group $\mathcal{G}_2$ of spinor relativity is the group of basis transformations, which relate spinor bases to each other; it is isomorphic to $\mathbf{GL}(\mathbb{C}^2)$:

$$\mathcal{G}_2 = \{(K_r, K_l) \in \mathbf{GL}(\mathbb{C}^2) \times \mathbf{GL}(\mathbb{C}^2) \mid K_l^+ K_r = \mathbf{1}_2\} \cong \mathbf{GL}(\mathbb{C}^2) \tag{1.16}$$

The gauge group acts on sections of the tangent bundle in a spinor basis in form of right and left spinor representations of the Lorentz group (together with $\mathbf{U}(1)$ and $\mathbf{D}(1)$ transformations), whence these sections are called *spinor fields*.

Despite the apparent analogy between orthonormal tetrad bases in general relativity and spinor bases in spinor relativity, there is an important difference concerning quantum theory. According to (1.12) a quantized metric implies that the tetrad field also must be quantized. However, it is always possible to adapt the orthonormal bases to the coordinates such that corresponding to (1.2) the tetrad fields have unit vacuum expectation value:

$$\langle e^\mu{}_\alpha \rangle_0 = \delta^\mu{}_\alpha \tag{1.17}$$

This means that the transformation from coordinates to orthonormal bases on spacetime is essentially a classical transformation with only small quantum corrections. In contrast to the spacetime metric, the Dirac metric is assumed to fluctuate strongly and is not required to have a simple ground state expectation value. According to (1.15) it is consequently not possible to choose a spinor basis such that both dyad fields have unit ground state expectation value. This means that the transformation from coordinates to spinor bases on a chiral manifold is a *quantum basis transformation*, which cannot be interpreted classically. A similar quantum basis transformation relates spinor fields and vector fields, as described below.

The Dirac metric gives rise to a 2-form in analogy to the Kähler form on a hermitian manifold. This form is further required to satisfy the *chiral Kähler condition*

$$\bar{d}_{ch}\hat{g} = 0, \quad \hat{g} = i g_{\dot{M}N}\, dz_l^{\dot{M}} \wedge dz_r^N \tag{1.18}$$

where the chiral differential $d_{ch} = \partial^r + \bar{\partial}^l$ is the sum of the holomorphic differential on the right manifold and the antiholomorphic differential on the left manifold. A chiral manifold equipped with a complex structure commuting with the chiral structure and a Dirac metric satisfying the chiral Kähler condition is called a *Dirac manifold*. A covariant derivative on a Dirac manifold is provided by the *Dirac connection*. It is defined by its right and left connection matrices, which are given in terms of the Dirac metric as follows

$$\Theta_r{}^M{}_N = g^{M\dot{S}}\partial g_{\dot{S}N}, \qquad \Theta_l{}^M{}_N = g^{+M\dot{S}}\partial g^+_{\dot{S}N} \tag{1.19}$$

where the inverse Dirac metric is denoted by upper indices. In general relativity the Levi-Civita connection on spacetime is determined uniquely by the two conditions of compatibility with the metric and vanishing torsion. The Dirac connection is also uniquely determined by two conditions, the first being compatibility with the Dirac metric, while the second requires the components of the connection matrices to be differential forms of type $(1, 0)$. The torsion of the Dirac connection does in general not vanish,[3] which is essential for spinor relativity, since the contracted torsion provides the *fundamental spinor field* Ψ:

$$\Psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix}, \qquad \chi^+_A = T_r{}^B{}_{BA}, \qquad \varphi^+_A = T_l{}^B{}_{BA} \tag{1.20}$$

In analogy to general relativity a dynamical equation for the Dirac metric is obtained from an action principle with action functional given by the integral over spacetime of a real curvature scalar density. As a consequence of the chiral Kähler condition, the Dirac metric may not be varied freely on the entire chiral manifold but only on a four-dimensional submanifold. The requirement of stationary action yields the *spinor Einstein equation*

$$R_{\dot{A}B} - R g_{\dot{A}B} = \chi^+_B \varphi_{\dot{A}} - \mathrm{D}^r_B \varphi_{\dot{A}} \tag{1.21}$$

where the *Ricci spinor* and curvature scalar on its left hand side are defined as contractions of the right curvature spinor:

$$R_{\dot{A}B} = R_r{}^C{}_{B\dot{A}C}, \qquad R = R^A{}_A \tag{1.22}$$

In contrast to pure general relativity, where the Einstein tensor is required to vanish, a 'matter term' involving the spinor field and its covariant derivative arises on the right hand side of the spinor Einstein equation in a natural way.

[3]The Dirac metric may be considered as an indefinite hermitian metric on the eight-dimensional manifold M_{ch} of the special form $\left(\begin{smallmatrix} 0 & g^+ \\ g & 0 \end{smallmatrix}\right)$. The Dirac connection is then a special case of the canonical connection on a hermitian holomorphic vector bundle [3, 4]. In case of a Kähler manifold, i.e. if the metric satisfies the (non-chiral) Kähler condition $d\mathrm{Re}\,\hat{g} = 0$, the torsion vanishes and the canonical connection is equal to the Levi-Civita connection of the underlying eight-dimensional pseudo-riemannian metric. The chiral Kähler condition is less restrictive, allowing for a non-vanishing torsion.

The fundamental spinor field has a two-fold significance in spinor relativity. On the one hand it is directly interpreted as a physical spinor field, on the other hand it allows to define distinguished basis transformations in the tangent bundle of a Dirac manifold, which associate a *Minkowski basis* to a given spinor basis. These basis transformations are accomplished by *spinor tetrad fields* ξ and ζ, which carry a spinor and a vector index and are defined in terms of the fundamental spinor field and Weyl matrices by the expressions:

$$\xi_A^\alpha = \frac{(\varphi^+\sigma^\alpha)_A}{\sqrt{\varphi^+\chi}}, \qquad \zeta_A^\alpha = \frac{(\chi^+\hat{\sigma}^\alpha)_A}{\sqrt{\chi^+\varphi}} \tag{1.23}$$

A section of the tangent bundle in a Minkowski basis is called a *vector field*. Its components are related to the spinor field components as follows:

$$V^\alpha = \xi_A^\alpha \phi_r^A + \zeta_{\dot{A}}^\alpha \bar{\phi}_l^{\dot{A}}; \quad \phi_r^A = \frac{1}{2}\xi_\alpha^A V^\alpha, \ \phi_l^A = \frac{1}{2}\zeta_\alpha^A \bar{V}^\alpha \tag{1.24}$$

The Minkowski bases may further be decomposed into their real and imaginary parts, defining respectively the four-dimensional subspaces of *classical* and *axial vectors*. $\mathbf{SL}(\mathbb{C}^2)$-gauge transformations are represented on vector fields by real Lorentz transformations, while electromagnetic phase transformations are represented trivially. The decomposition of the tangent bundle into classical and axial vector fields is however not invariant, since $\mathbf{D}(1)$-gauge transformations mix them. The inner product in the tangent bundle takes the form of the Minkowski metric on classical vectors, justifying the name of Minkowski bases.

Minkowski bases do not respect the complex and chiral structures of the Dirac manifold, these are represented by non-constant fields

$$\begin{array}{llll} \text{Complex structure:} & \phi_r^A \to i\phi_r^A, \quad \phi_l^A \to i\phi_l^A & \Rightarrow & V^\alpha \to I^\alpha{}_\beta V^\beta \\ \text{Chiral structure:} & \phi_r^A \to \phi_r^A, \quad \phi_l^A \to -\phi_l^A & \Rightarrow & V^\alpha \to -iI^\alpha{}_\beta V^\beta \end{array} \tag{1.25}$$

where the tensor I, which represents the complex structure, is defined by:

$$I_{\alpha\beta} = i\delta_{\dot{A}B}\zeta_{[\alpha}^{\dot{A}}\xi_{\beta]}^B \tag{1.26}$$

The relations (1.25) further show that Minkowski bases are compatible with the product of complex and chiral structure, which is represented by i. This means that the imaginary units in complex vectors and in spinors refer to different complex structures. This circumstance may be accounted for by introducing a 'bicomplex formalism', where the chiral structure is expressed in terms of a chiral unit j with $j^2 = 1$. In this formalism[4] the imaginary unit in complex vectors is ij and Dirac spinors take the form of 'bicomplex' two-component spinors with chiral projectors $\frac{1}{2}(1 \pm j)$.

The transformation from a spinor basis to a Minkowski basis is a quantum basis transformation, since the spinor tetrad fields have vanishing ground state expectation values as a consequence of Lorentz invariance

$$\langle \xi_A^\alpha \rangle_0 = 0, \qquad \langle \zeta_A^\alpha \rangle_0 = 0 \tag{1.27}$$

and are further supposed to have large quantum fluctuations. This has characteristic consequences concerning the nature of fields as seen by a classical observer and explains why

[4] The bicomplex formalism will not be used in this paper.

spinor fields on spacetime do not appear as sections of the tangent bundle but seem to be related to the latter only via the Clifford bundle. A classical observer is related to a local inertial system in its neighborhood, and one may introduce a Minkowski basis closely adapted to the approximate inertial coordinates. From the point of view of such an observer a section of the tangent bundle of the Dirac manifold appears as a tangent vector field, if its components with respect to the adapted Minkowski basis behave classically with small fluctuations. On the other hand, a section with slowly varying spinor basis components, describing e.g. the wavefunction of an electron in an atom, according to (1.24) and (1.27) has Minkowski basis components with vanishing expectation values and large quantum fluctuations and will not appear as a tangent vector field from the point of view of the classical observer.

The spinor Einstein equation may be transformed to Minkowski bases, where it decomposes into *electromagnetic* and *gravitational Einstein equations:*

$$\begin{aligned} \mathcal{F}_{\alpha\beta} &= \frac{1}{2}(\mathrm{D}_\gamma J_\delta)(\eta^\gamma{}_{[\alpha}\eta^\delta{}_{\beta]} + I^\gamma{}_{[\alpha}I^\delta{}_{\beta]}) + \frac{1}{4}\mu^2 \mathrm{Re}\, I_{\alpha\beta} + \frac{1}{4}\sigma I_{\alpha\beta} \\ \mathcal{G}_{\alpha\beta} &= -\frac{i}{2}(\mathrm{D}_\gamma T_\delta)(\eta^\gamma{}_\alpha\eta^\delta{}_\beta + I^\gamma{}_\alpha I^\delta{}_\beta) + \frac{1}{4}\mu^2(I\bar{I}+\eta)_{\alpha\beta} \end{aligned} \tag{1.28}$$

$\mathcal{G}$ denotes the Einstein tensor of the Dirac connection while the complex electromagnetic field $\mathcal{F}$ combines the $\mathbf{U}(1)$ electromagnetic field F and the $\mathbf{D}(1)$ axial field $\widetilde{F}$ as its real and imaginary part respectively. The *current vector* J, the *torsion vector* T and the scalars μ and σ on the right hand sides of these equations are defined in terms of the fundamental spinor field as follows:

$$J_\alpha = \frac{i}{2}\bar{\mu}^{-1}\bar{\Psi}\gamma_\alpha\Psi, \qquad T_\alpha = \frac{i}{2}\bar{\mu}^{-1}\bar{\Psi}\gamma_5\gamma_\alpha\Psi, \qquad \mu^2 = \chi^+\varphi, \qquad i\sigma = \mathrm{D}^r_A\varphi^A \tag{1.29}$$

The Einstein equations involve only covariant derivatives D_α in the directions of classical basis vectors, since derivatives in axial basis vector directions may be eliminated as a consequence of the chiral Kähler condition.

Besides the Dirac connection D, which has non-vanishing torsion, the *Einstein connection* ∇ is introduced, which is obtained from removing the torsion from the Dirac connection. This connection becomes equal to the Levi-Civita connection of the Lorentz metric on spacetime in an approximation valid in the vicinity of the ground state and thus describes the gravitational field. Written in terms of the Einstein connection (1.28) takes the form

$$\begin{aligned} \mathcal{F}_{\alpha\beta} &= \frac{1}{2}\nabla_{[\alpha}J_{\beta]} + \frac{i}{4}\epsilon_{\alpha\beta}{}^{\gamma\delta}\nabla_\gamma J_\delta + \frac{1}{4}\mu^2\mathrm{Re}\, I_{\alpha\beta} \\ G_{\alpha\beta} &= \frac{1}{2}\bar{\mu}^{-1}[\bar{\Psi}\overset{\leftrightarrow}{\nabla}_{(\alpha}\gamma_{\beta)}\Psi] + T_\alpha T_\beta - \frac{1}{4}\mu^2(I\bar{I})_{\alpha\beta} - \frac{1}{2}(\sigma+\mu^2)\eta_{\alpha\beta} + \{\widetilde{A}, \mathrm{Im}\,\Gamma, \partial\mu\} \end{aligned} \tag{1.30}$$

where G denotes the Einstein tensor of ∇. The Dirac derivative of the torsion vector on the right hand side of the gravitational Einstein equation has been reexpressed in terms of Einstein derivatives of the fundamental spinor field. The curly brackets indicate terms involving the axial vector potential $\widetilde{A}$ and the imaginary part of the Einstein connection coefficients Γ, which arise from covariant derivatives of the Dirac matrices, as well as terms involving derivatives of the scalar field μ. In the vicinity of the ground state these terms vanish and the Dirac matrices become covariantly constant with respect to the Einstein connection.

One may further derive complex *Maxwell equations*. While the homogeneous equation immediately follows from the Bianchi identity, the inhomogeneous equation is obtained from taking the covariant divergence of the electromagnetic Einstein equation and simplifying the result with help of the Bianchi identity:

$$\nabla_\beta \mathcal{F}^{\alpha\beta} = -\frac{1}{4}\mu^2 J^\alpha + \frac{1}{4} I^{\alpha\beta}\partial_\beta \mu^2, \qquad \nabla_\beta \mathcal{F}^{*\alpha\beta} = 0 \tag{1.31}$$

Finally, the fundamental spinor field satisfies an identity independent of the dynamical spinor Einstein equation, which takes the form of a generalized non-linear Dirac equation and is called the *Dirac identity:*

$$\gamma^\alpha \nabla_\alpha \Psi + \frac{1}{4}[(\mathrm{Im} + i\gamma_5 \mathrm{Re})T_\alpha]\gamma^\alpha \Psi = \{\widetilde{A}, \mathrm{Im}\,\Gamma, \partial\mu\} \tag{1.32}$$

Again, the curly brackets indicate terms, which vanish in the vicinity of the ground state.

The first terms on the right hand sides of the gravitational Einstein equation (1.30) and the inhomogeneous Maxwell equation (1.31) take the form of the Belinfante energy-momentum tensor and current density respectively of a Dirac spinor field, but multiplied with a scalar field instead of a constant. This motivates the assumption that the invariant square of the fundamental spinor field have a non-vanishing ground state expectation value:

$$\langle \mu^2 \rangle_0 = \mu_0^2 > 0 \quad \Leftrightarrow \quad \langle \bar{\Psi}\Psi \rangle_0 = 2\mu_0^2, \qquad \langle \bar{\Psi}\gamma_5\Psi \rangle_0 = 0 \tag{1.33}$$

The ground state is further required to break $\mathbf{D}(1)$-gauge symmetry, which is accomplished by the assumption of a non-vanishing ground state expectation value for products of right or left spinor fields with dyad fields as follows:

$$\langle \varphi^A \varphi_B^+ E_r^C{}_M E_l^M{}_D \rangle_0 = \frac{1}{2}\mu_0^2 \epsilon_r^{AC} \epsilon_{BD}^l, \qquad \langle \chi^A \chi_B^+ E_l^C{}_M E_r^M{}_D \rangle_0 = \frac{1}{2}\mu_0^2 \epsilon_l^{AC} \epsilon_{BD}^r \tag{1.34}$$

Since the chiral coordinate indices on the right and left dyad fields in these expressions refer to the right and left manifold respectively, their contraction in products of dyad fields breakes the chiral symmetry of independent coordinate transformations on the right and left manifold. The ground state thus distinguishes a class of coordinate systems, which are compatible with (1.34). These may further be used to define a preferred four-dimensional submanifold $\mathcal{S}$ by the condition

$$z_r^M|_{\mathcal{S}} = z_l^M|_{\mathcal{S}} = z^M \tag{1.35}$$

which is required to hold for chiral coordinates compatible with (1.34) and which is left invariant under the unbroken symmetry of joint coordinate transformations on the right and left manifold. This submanifold is identified with spacetime.

In order to investigate the implications of the assumptions (1.33) and (1.34) systematically, one might use non-linear realizations of the broken symmetries [5, 6]. Instead, a very simple approximation is used here, which consists in replacing the expressions on the left hand sides of (1.33) and (1.34) with their ground state expectation values and is called the *vacuum approximation*. Since μ_0^2 will be seen to be of Planck order, fields at ordinary energies may be considered as small excitations above the ground state, and the vacuum approximation should be appropriate for their description.

The $\mathbf{D}(1)$-symmetry breaking of the ground state causes a decomposition of the tangent spaces into classical and axial vectors, which is invariant under the unbroken gauge group of Lorentz and electromagnetic phase transformations. In the vacuum approximation the spaces of classical vectors become equal to the tangent spaces of spacetime, which may be seen as follows. In general, the commutators of classical basis vectors e_α are given by

$$[e_\alpha, e_\beta] = \mathrm{Re}\, c^\gamma{}_{[\alpha\beta]} e_\gamma - \mathrm{Im}\, c^\gamma{}_{[\alpha\beta]} \tilde{e}_\gamma \tag{1.36}$$

where $\tilde{e}_\alpha$ denotes axial basis vectors. In the vacuum approximation the structure functions become real

$$\mathrm{Im}\, c^\gamma{}_{\beta\alpha} \overset{\circ}{=} 0 \tag{1.37}$$

where the symbol $\overset{\circ}{=}$ denotes equality within the vacuum approximation. This means that the spaces of classical vectors become integrable, i.e. they constitute the tangent spaces of a submanifold. This submanifold is spacetime, as may be seen from considering the derivatives ∂_α in the directions of classical basis vectors, which take the approximate form

$$\partial_\alpha \overset{\circ}{=} \mathrm{Re}[\xi_\alpha^M (\partial_M^r + \partial_M^l)] \tag{1.38}$$

where ∂_M^r and ∂_M^l denote derivatives with respect to the right and left coordinates z_r^M and z_l^M. The sum of these derivatives applied to a function on spacetime is equal to the derivative with respect to the joint coordinate z^M, i.e. ∂_α is a tangent derivative on $\mathcal{S}$.

Since the inner product in the tangent bundle takes Minkowski form on classical basis vectors, spacetime becomes equipped with a Lorentz metric, with respect to which the basis vectors e_α are orthonormal. As a consequence of (1.37) the connection coefficients Γ also become real

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(c_{\gamma[\alpha\beta]} + c_{\beta[\alpha\gamma]} - c_{\alpha[\beta\gamma]}) \quad \Rightarrow \quad \mathrm{Im}\, \Gamma^\alpha{}_{\beta\gamma} \overset{\circ}{=} 0 \tag{1.39}$$

and the Einstein connection is identified with the Levi-Civita connection on spacetime.

In order to compare the equations of spinor relativity in the vacuum approximation with the usual Einstein and Maxwell equations on spacetime, the dimensionless fields of spinor relativity must be equipped with their physical dimensions by multiplying them with suitable powers of a scale λ. The values of μ_0 and λ are then determined by the condition that the factors multiplying the Dirac current in the inhomogeneous Maxwell equation and the Belinfante energy-momentum tensor in the gravitational Einstein equation take the values $4\pi\alpha$ and $8\pi\lambda_P^2$ respectively, where α is the fine structure constant and λ_P the Planck length. This condition is satisfied if μ_0 and λ are chosen as follows:

$$\frac{1}{2}\langle\bar{\Psi}\Psi\rangle_0 = \mu_0^2 = \frac{1}{4\pi}\sqrt{\alpha}\lambda_P^{-3}, \qquad \lambda = 16\pi\sqrt{\alpha}\lambda_P \tag{1.40}$$

The first of these relations may be interpreted as the density of fermion-antifermion pairs in the ground state condensate.

Applying the vacuum approximation to the Maxwell equations (1.31) one obtains separate equations for the electromagnetic and axial fields:

$$\nabla_\beta F^{\alpha\beta} \overset{\circ}{=} -4\pi\alpha j^\alpha, \qquad \nabla_\beta F^{*\alpha\beta} \overset{\circ}{=} 0; \qquad \nabla_\beta \widetilde{F}^{\alpha\beta} \overset{\circ}{=} 0, \qquad \nabla_\beta \widetilde{F}^{*\alpha\beta} \overset{\circ}{=} 0 \tag{1.41}$$

The electromagnetic field is generated by the Dirac current

$$j^\alpha = i\bar{\Psi}\gamma^\alpha\Psi \tag{1.42}$$

while the axial field is source-free. This does however not mean that the axial field completely vanishes, as shown by the electromagnetic Einstein equation, which in the vacuum approximation takes the form:

$$F_{\alpha\beta} \stackrel{\circ}{=} i\pi\sqrt{\alpha}\lambda_P\bar{\Psi}[\gamma_\alpha,\gamma_\beta]\Psi + 2\pi\lambda_P^2\nabla_{[\alpha}j_{\beta]}, \qquad \widetilde{F}_{\alpha\beta} \stackrel{\circ}{=} \pi\lambda_P^2\epsilon_{\alpha\beta}{}^{\gamma\delta}\nabla_\gamma j_\delta \tag{1.43}$$

The terms arising from the rotation of the current are suppressed by a factor of λ_P^2 and the axial field is thus very small. The main part of the electromagnetic field F is given by the first term on the right hand side of (1.43), which is of the form (1.11) and may be interpreted as describing the polarization of the fermion-antifermion pairs, which constitute the ground state condensate.

Neglecting the contribution from the rotation of the current, one obtains a simple expression for the energy-momentum tensor of the electromagnetic field from (1.43):

$$T^{em}_{\alpha\beta} = F_{\alpha\gamma}F_\beta{}^\gamma - \frac{1}{4}\eta_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} \stackrel{\circ}{=} -\frac{1}{2}\alpha^2\lambda_P^{-4}(I\bar{I})_{\alpha\beta} \tag{1.44}$$

This shows that the third term on the right hand side of the gravitational Einstein equation (1.30) becomes proportional to the energy-momentum tensor of the electromagnetic field in the vacuum approximation and is moreover multiplied with the correct factor[5]:

$$G_{\alpha\beta} \stackrel{\circ}{=} 8\pi\lambda_P^2\left\{\frac{1}{2}\bar{\Psi}\overset{\leftrightarrow}{\nabla}_{(\alpha}\gamma_{\beta)}\Psi - 2\pi\lambda_P^2(\bar{\Psi}\gamma_5\gamma_\alpha\Psi)(\bar{\Psi}\gamma_5\gamma_\beta\Psi) + (4\pi\alpha)^{-1}T^{em}_{\alpha\beta}\right\} \tag{1.45}$$

The terms in curly brackets on the right hand side of (1.30) as well as the fourth term involving the scalar fields σ and μ vanish in the vacuum approximation. The source of the gravitational field becomes real as consistency with (1.39) requires.

The vacuum approximation of the Dirac identity (1.32) yields a generalized trilinear Dirac equation:

$$\gamma^\alpha\nabla_\alpha\Psi + \pi\lambda_P^2(\bar{\Psi}\gamma^\alpha\Psi)\gamma_\alpha\Psi \stackrel{\circ}{=} 0 \tag{1.46}$$

An equation of this type has been used by Heisenberg in his unified field theory [1] and also by Nambu and Jona-Lasinio in their work on chiral symmetry breaking [7].

Since the constant which appears in (1.46) is also of Planck order, the question arises, how the mass scale of ordinary particles may enter the theory. This question is related to an other problem of spinor relativity: In the classical theory spinors are identified with derivatives on a manifold and thus commute. In a quantized version of the theory one would like to replace the classical derivatives with anticommuting operators. This means that spinor relativity should not be quantized as a field theory on the Dirac manifold, but one should try to replace this manifold with a suitable non-commutative space, based on a fermionic quantum algebra of 'elementary events'. Such an approach also opens the possibility of introducing a fundamental length into the definition of the quantum derivative operators, providing a second length scale besides the Planck scale of the ground state expectation values.

2 Hermitian Manifolds

In this chapter the definitions of complex and hermitian manifolds as well as of connections on these manifolds are shortly reviewed in a way suitable for generalization to Dirac manifolds in the next chapter. As a particularly simple and natural connection on a hermitian

[5] The electromagnetic field is normalized such that no coupling constant appears in the covariant derivative.

manifold the canonical connection [3, 4] is introduced and its curvature and torsion are investigated. The emphasis in this and the following chapters is entirely on local properties of the connection, which in analogy to general relativity is considered as a physical field. Global properties of the manifolds are not taken into account here; in particular, one may always assume the manifolds to have trivial topology.

2.1 Complex Manifolds

A vector space $V \cong \mathbb{R}^{2n}$ is said to have a *complex structure*, if there exists an automorphism I such that:

$$I^2 = -\mathbf{1}_{2n} \tag{2.1}$$

The subalgebra of the endomorphisms of V which leave this complex structure invariant, i.e. which commute with I, is isomorphic to the algebra of $n \times n$-matrices with complex elements:

$$\mathbf{AL}(\mathbb{C}^n) \cong \{K \in \mathbf{AL}(\mathbb{R}^{2n}) \mid [I, K] = 0\} \tag{2.2}$$

Since I commutes with all elements of $\mathbf{AL}(\mathbb{C}^n)$, it is represented on $\mathbb{C}^n$ by a matrix $\pm i\mathbf{1}_n$ proportional to the unit matrix. The two possibilities to choose the sign in this matrix correspond to two vector spaces W, $\bar{W} \cong \mathbb{C}^n$ into which $\mathbb{C} \otimes V$ decomposes, with I given by:

$$W: \ I = +i\mathbf{1}_n, \qquad \bar{W}: \ I = -i\mathbf{1}_n \tag{2.3}$$

Complex conjugation maps these spaces bijectively onto each other. The space $V^*_{\mathbb{C}} = \mathbb{C} \otimes V^*$ of complex valued $\mathbb{R}$-linear forms over V decomposes into the direct sum $V^*_{\mathbb{C}} = W^* \oplus \bar{W}^*$ of the dual spaces of W and $\bar{W}$.

A *complex manifold* M of dimension n is a real $2n$-dimensional manifold together with a smooth complex structure in its tangent bundle $V(M)$, which is further required to satisfy the integrability condition

$$I^\nu{}_\rho \partial_{[\sigma} I^\mu{}_{\nu]} - I^\nu{}_\sigma \partial_{[\rho} I^\mu{}_{\nu]} = 0 \tag{2.4}$$

where the indices refer to arbitrary real coordinates on the manifold. This integrability condition allows to introduce complex coordinates $\{z^M\}_{M \in \{1\ldots n\}}$ such that the complex bundles $W(M)$ and $\bar{W}(M)$ associated with $V(M)$ are spanned by the bases $\{\partial_M\}_{M \in \{1\ldots n\}}$ and $\{\partial_{\dot{M}}\}_{\dot{M} \in \{1\ldots n\}}$ respectively, where ∂_M denotes partial derivative with respect to the coordinate z^M and $\partial_{\dot{M}}$ partial derivative with respect to the complex conjugate coordinate $z^{\dot{M}}$.

The bundle $V^*_{\mathbb{C}}(M)$ of complex-valued 1-forms on the manifold decomposes into the sum of the bundles $W^*(M)$ and $\bar{W}^*(M)$, the sections of which are called forms of type $(1, 0)$ and of type $(0, 1)$ respectively. Extending this decomposition to the antisymmetric tensor products $\bigwedge^r V^*_{\mathbb{C}}(M)$, an r-form on a complex manifold is the sum of forms of type (p, q) with $p + q = r$. The exterior derivative $d\alpha$ of a form α of type (p, q) is the sum of a form $\partial\alpha$ of type $(p + 1, q)$ and a form $\bar{\partial}\alpha$ of type $(p, q + 1)$, where ∂ and $\bar{\partial}$ are the holomorphic and antiholomorphic differential respectively:

$$\partial = a^+(dz^M)\partial_M, \qquad \bar{\partial} = a^+(dz^{\dot{M}})\partial_{\dot{M}}; \qquad d = \partial + \bar{\partial}, \qquad \partial^2 = \bar{\partial}^2 = \{\partial, \bar{\partial}\} = 0 \tag{2.5}$$

$a^+(\beta)$ with $\beta \in V^*_{\mathbb{C}}(M)$ denotes an operator, which acts on differential forms by exterior multiplication with its argument, $a^+(\beta)\alpha := \beta \wedge \alpha$.

On a complex manifold the structure preserving coordinate transformations are those, where (locally) the new coordinates are holomorphic functions of the old coordinates. The basis transformations in the tangent bundle induced by holomorphic coordinate transformations leave $W(M)$ and $\bar{W}(M)$ invariant.

2.2 Connections on a Complex Manifold

A *connection* ∇ in the tangent bundle of a complex manifold is a $\mathbb{C}$-linear map satisfying:

$$\begin{aligned} &\nabla : W(M) \to V^*_{\mathbb{C}}(M) \otimes W(M) \\ &\nabla(f\psi) = df \otimes \psi + f\nabla\psi; \quad \psi \in W(M), f \in C(M) \end{aligned} \tag{2.6}$$

This definition is extended from $W(M)$ to the associated bundles $\bar{W}(M)$, $W^*(M)$ and $\bar{W}^*(M)$ by the requirements of reality and of compatibility with index contraction. The Leibniz rule then allows further extension of the connection to arbitrary tensor products of the tangent and cotangent bundles.

A connection is completely determined by its *connection matrix* Θ, which is a matrix of 1-forms defined by the action of ∇ on a basis of the tangent bundle

$$\nabla e_A = e_B \Theta^B{}_A, \qquad \nabla \omega^A = -\Theta^A{}_B \omega^B, \qquad \nabla e_{\dot{A}} = e_{\dot{B}} \bar{\Theta}^{\dot{B}}{}_{\dot{A}}, \qquad \nabla \omega^{\dot{A}} = -\bar{\Theta}^{\dot{A}}{}_{\dot{B}} \omega^{\dot{B}} \tag{2.7}$$

where $\{e_A\}_{A\in\{1\dots n\}}$ denotes a basis of $W(M)$ and $\{\omega^A\}$ the basis of $W^*(M)$ dual to $\{e_A\}$; the corresponding bases $\{e_{\dot{A}}\}$ and $\{\omega^{\dot{A}}\}$ of $\bar{W}(M)$ and $\bar{W}^*(M)$ respectively are obtained by complex conjugation of the former bases.

The defining properties of a connection require an inhomogeneous transformation behaviour of the connection matrix under basis transformations $e_{A'} = e_B K^{-1B}{}_{A'}$ in the tangent bundle:

$$\Theta^{A'}{}_{B'} = K^{A'}{}_C \Theta^C{}_D K^{-1D}{}_{B'} + K^{A'}{}_C dK^{-1C}{}_{B'} \tag{2.8}$$

The *curvature matrix* Ω of the connection is defined by:

$$\Omega^A{}_B = d\Theta^A{}_B + \Theta^A{}_C \wedge \Theta^C{}_B \tag{2.9}$$

It is a matrix of 2-forms which transforms homogeneously under basis transformations. Exterior derivation of (2.9) yields the *first Bianchi identity:*

$$d\Omega^A{}_B + \Theta^A{}_C \wedge \Omega^C{}_B - \Omega^A{}_C \wedge \Theta^C{}_B = 0 \tag{2.10}$$

The *torsion form* τ of the connection is defined as follows:

$$\tau^A = d\omega^A + \Theta^A{}_B \wedge \omega^B \tag{2.11}$$

It is a vector of 2-forms, which transforms homogeneously under basis transformations. Exterior derivation of (2.11) yields the *second Bianchi identity:*

$$d\tau^A + \Theta^A{}_B \wedge \tau^B = \Omega^A{}_B \wedge \omega^B \tag{2.12}$$

2.3 Canonical Connection

A *hermitian manifold* is a complex manifold equipped with a symmetric bilinear product in its tangent bundle, which takes non-zero values between sections of $W(M)$ and $\bar{W}(M)$ only:

$$(e_{\dot{A}}, e_B) = (e_B, e_{\dot{A}}) = h_{\dot{A}B}, \qquad (e_A, e_B) = (e_{\dot{A}}, e_{\dot{B}}) = 0; \qquad h^{+}_{\dot{A}B} = h_{\dot{A}B} \tag{2.13}$$

The matrix $h_{\dot{A}B}$ is called the *hermitian metric* and gives rise to a real 2-form of type $(1, 1)$, which is called the *Kähler form:*

$$\hat{h} = i h_{\dot{M}N} dz^{\dot{M}} \wedge dz^{N} \tag{2.14}$$

On a hermitian manifold a particularly simple connection may be defined in terms of the hermitian metric, which is called the *canonical connection*. The connection matrix of the canonical connection is a matrix of forms of type $(1, 0)$ given by

$$\Theta^{M}{}_{N} = h^{M\dot{S}} \partial h_{\dot{S}N} \tag{2.15}$$

where the inverse of the hermitian metric is denoted with upper indices. In (2.15) indices from the middle of the alphabet have been used in order to indicate that this expression for the connection matrix is valid in a coordinate basis (more general a holomorphic basis) only. As may be seen from (2.8) the canonical connection matrix retains its form (2.15) under a basis transformation in the tangent bundle only, if the transformation matrix satisfies $\bar{\partial} K = \partial \bar{K} = 0$, as is the case for holomorphic coordinate transformations. In a general[6] basis the connection matrix takes the form

$$\Theta^{A}{}_{B} = E^{A}{}_{M} \Theta^{M}{}_{N} E^{N}{}_{B} + E^{A}{}_{M} d E^{M}{}_{B} \tag{2.16}$$

where $E^{A}{}_{M}$ denotes the transformation matrix from a coordinate basis to a general basis in $W(M)$ and $E^{M}{}_{A}$ its inverse. Since in general $\bar{\partial} E$ is non-zero, the connection matrix acquires a (0, 1)-type part in non-holomorphic bases.

The canonical connection could alternatively be defined to be the unique connection satisfying the following two requirements:

∇ is compatible with the hermitian metric:

$$d(e_{\dot{A}}, e_B) = (\nabla e_{\dot{A}}, e_B) + (e_{\dot{A}}, \nabla e_B) \quad \Leftrightarrow \quad d h_{\dot{A}B} = h_{\dot{A}C} \Theta^{C}{}_{B} + \bar{\Theta}^{\dot{C}}{}_{\dot{A}} h_{\dot{C}B} \tag{2.17}$$

$\Theta^{M}{}_{N}$ is a matrix of 1-forms of type $(1, 0)$ for coordinate bases.

An important class of non-holomorphic bases are the *unitary bases*, which are defined by the requirement that the hermitian metric be equal to the unit matrix:

$$\bar{E}^{\dot{M}}{}_{\dot{A}} h_{\dot{M}N} E^{N}{}_{B} = \delta_{\dot{A}B} \tag{2.18}$$

As indicated by their name, the group of transformations, which transform unitary bases into each other, is the unitary group:

$$\mathbf{U}(n) = \{K \in \mathbf{GL}(\mathbb{C}^n) | K^{+} K = \mathbf{1}_n\} \tag{2.19}$$

[6] A 'general basis' in the tangent bundle is not completely arbitrary, since it is required to be compatible with the complex structure.

In a unitary basis the connection matrix (with first index lowered with the metric) is antihermitian, as follows from the compatibility of the connection with the metric (2.17):

$$\Theta^{+}_{\dot{A}B} = -\Theta_{\dot{A}B} \tag{2.20}$$

2.4 Curvature and Torsion of the Canonical Connection

The curvature matrix of the canonical connection is a matrix of 2-forms of type $(1,1)$, since the $(2,0)$-type terms on the right hand side of (2.9) vanish identically, as may be seen explicitly using a coordinate basis:

$$\Omega^{M}{}_{N} = \bar{\partial}\Theta^{M}{}_{N}, \qquad \partial\Theta^{M}{}_{N} + \Theta^{M}{}_{S} \wedge \Theta^{S}{}_{N} = 0 \tag{2.21}$$

As a consequence of the compatibility of the connection with the metric the curvature matrix is antihermitian:

$$\Omega^{+}_{\dot{A}B} = -\Omega_{\dot{A}B} \tag{2.22}$$

This may also be seen from its expression in terms of the hermitian metric:

$$\Omega_{\dot{M}N} = h_{\dot{M}S}\Omega^{S}{}_{N} = \bar{\partial}\partial h_{\dot{M}N} - (\bar{\partial}h_{\dot{M}S}) \wedge h^{S\dot{R}}\partial h_{\dot{R}N} \tag{2.23}$$

An expansion of the curvature 2-forms in terms of a basis in the cotangent bundle defines the *curvature tensor:*

$$\Omega^{A}{}_{B} = R^{A}{}_{B\dot{C}D}\, \omega^{\dot{C}} \wedge \omega^{D} \tag{2.24}$$

As a consequence of (2.22) the curvature tensor satisfies the following symmetry property:

$$R_{\dot{A}B\dot{C}D} = \bar{R}_{B\dot{A}D\dot{C}} \tag{2.25}$$

In contrast to the Riemann tensor in general relativity, which may be contracted in a unique way (up to sign), yielding the Ricci tensor and the curvature scalar, there are several possibilities of contracting the curvature tensor of the canonical connection; one obtains four hermitian *Ricci type tensors* and two real *curvature scalars:*

$$\begin{aligned} R^{(1)}_{\dot{A}B} &= \frac{1}{2}(R^{C}{}_{B\dot{A}C} + R_{\dot{A}C}{}^{C}{}_{B}), \qquad \rho_{\dot{A}B} = R^{C}{}_{C\dot{A}B}, \qquad R = R^{B}_{(1)B} \\ R^{(2)}_{\dot{A}B} &= \frac{i}{2}(R^{C}{}_{B\dot{A}C} - R_{\dot{A}C}{}^{C}{}_{B}), \qquad \tilde{\rho}_{\dot{A}B} = R_{\dot{A}B}{}^{C}{}_{C}, \qquad \rho = \rho^{B}{}_{B} \end{aligned} \tag{2.26}$$

In a coordinate basis the expression (2.11) for the torsion form simplifies

$$\tau^{M} = \Theta^{M}{}_{N} \wedge dz^{N} \tag{2.27}$$

which shows that the torsion form of the canonical connection is a vector of 2-forms of type $(2,0)$. An expansion of these forms in terms of a basis in the cotangent bundle defines the *torsion tensor:*

$$\tau^{A} = \frac{1}{2}T^{A}{}_{BC}\, \omega^{B} \wedge \omega^{C} \tag{2.28}$$

A special class of hermitian manifolds are the *Kähler manifolds*. Their defining property is a closed Kähler form, which is equivalent to a vanishing torsion of the canonical connection:

$$d\hat{h} = \mathrm{Im}(T_{\dot{A}BC}\omega^{\dot{A}} \wedge \omega^B \wedge \omega^C) = 0 \tag{2.29}$$

On a Kähler manifold the canonical connection is equal to the Levi-Civita connection of the underlying riemannian manifold.

In the special case of a *hermitian manifold of dimension* $n = 2$ there is a linear relation between the Ricci type tensors

$$2R^{(1)}_{\dot{A}B} - \rho_{\dot{A}B} - \tilde{\rho}_{\dot{A}B} + (\rho - R)h_{\dot{A}B} = 0 \tag{2.30}$$

and the torsion may be entirely expressed in terms of the contracted torsion tensor:

$$T^A{}_{BC} = \delta^A{}_B T_C - \delta^A{}_C T_B, \qquad T_A = T^B{}_{BA} \tag{2.31}$$

The identity (2.30) may be derived with help of an expansion of the curvature and Ricci type tensors in terms of Pauli and unit matrices,[7] while (2.31) is verified immediately taking into account that in two dimensions the torsion tensor has only two independent components as a consequence of the antisymmetry of its last index pair.

3 Dirac Manifolds

The Dirac manifolds considered in this chapter are complex manifolds of dimension $2n$, which in addition carry a 'chiral structure', which decomposes them into a product of two manifolds of dimension n, which are called the right and left manifold. The Dirac manifolds are further equipped with a Dirac metric, which is a hermitian metric satisfying certain conditions with respect to the chiral structure. As a consequence of these conditions, the canonical connection derived from the Dirac metric effectively reduces to a pair of connections acting on the right and left manifold respectively.

3.1 Chiral Manifolds

A vector space $V \cong \mathbb{R}^{4n}$ with complex structure I is said to have a *chiral structure*, if there exists an automorphism J such that:

$$J^2 = \mathbf{1}_{4n}, \qquad [I, J] = 0, \qquad \mathrm{Tr} J = 0 \tag{3.1}$$

[7] Using a unitary basis, an expansion of the curvature tensor in terms of Pauli and unit matrices takes the form

$$R_{\dot{A}B\dot{C}D} = \frac{1}{4}[\rho\delta_{\dot{A}B}\delta_{\dot{C}D} + r_k\delta_{\dot{A}B}\sigma^k_{\dot{C}D} + \tilde{r}_k\sigma^k_{\dot{A}B}\delta_{\dot{C}D} + r_{mn}\sigma^m_{\dot{A}B}\sigma^n_{\dot{C}D}]$$

where the coefficients are real as a consequence of (2.25). Contraction with the metric then yields the following expressions for the Ricci type tensors from which the identity (2.30) follows immediately:

$$\rho_{\dot{A}B} = \frac{1}{2}[\rho\delta_{\dot{A}B} + r_k\sigma^k_{\dot{A}B}], \qquad \tilde{\rho}_{\dot{A}B} = \frac{1}{2}[\rho\delta_{\dot{A}B} + \tilde{r}_k\sigma^k_{\dot{A}B}],$$

$$R^{(1)}_{\dot{A}B} = \frac{1}{4}[(\rho + r_{kk})\delta_{\dot{A}B} + (r_k + \tilde{r}_k)\sigma^k_{\dot{A}B}], \qquad R^{(2)}_{\dot{A}B} = \frac{1}{4}\epsilon_{kmn}r_{mn}\sigma^k_{\dot{A}B}.$$

Since J commutes with the complex structure, it leaves the complex spaces W and $\bar{W}$ associated to V invariant. As a consequence of its vanishing trace, J has equal numbers of eigenvalues ± 1. Therefore W and $\bar{W}$ decompose into isomorphic subspaces

$$W = W_r \oplus W_l, \qquad \bar{W} = \bar{W}_r \oplus \bar{W}_l \tag{3.2}$$

where the right spaces W_r and $\bar{W}_r$ are eigenspaces of J for eigenvalue $+1$ and the left spaces W_l and $\bar{W}_l$ are eigenspaces of J for eigenvalue -1.

A *chiral manifold* M_{ch} of dimension n is a complex $2n$-dimensional manifold together with a smooth chiral structure in its tangent bundle, which is further required to satisfy the integrability condition

$$J^Q{}_R \partial_{[S} J^M{}_{Q]} - J^Q{}_S \partial_{[R} J^M{}_{Q]} = 0, \qquad \partial_{\dot{R}} J^M{}_S - J^Q{}_S \partial_{\dot{R}} J^M{}_Q = 0 \tag{3.3}$$

where the indices refer to arbitrary complex coordinates compatible with I. The integrability condition allows to introduce complex coordinates $\{z_r^M, z_l^M\}_{M\in\{1\ldots n\}}$ such that the right and left subbundles $W(M_r)$ and $W(M_l)$ into which the chiral structure decomposes $W(M_{ch})$ are spanned by the bases $\{\partial^r_M\}_{M\in\{1\ldots n\}}$ and $\{\partial^l_M\}_{M\in\{1\ldots n\}}$ respectively, where ∂^i_M with $i \in \{r, l\}$ denotes partial derivative with respect to the coordinate z_i^M.

The sections of the right and left cotangent bundles $W^*(M_r)$ and $W^*(M_l)$ are called forms of type $(1_r, 0)$ and $(1_l, 0)$ respectively, while the sections of $\bar{W}^*(M_r)$ and $\bar{W}^*(M_l)$ are forms of type $(0, 1_r)$ and $(0, 1_l)$. The holomorphic and antiholomorphic differentials also decompose into right and left parts:

$$\partial^i = a^+(dz_i^M)\partial^i_M, \qquad \bar{\partial}^i = a^+(dz_i^{\dot{M}})\partial^i_{\dot{M}}; \qquad \partial = \partial^r + \partial^l, \qquad \bar{\partial} = \bar{\partial}^r + \bar{\partial}^l \tag{3.4}$$

In the following the sum of subbundles which are related to each other by complex conjugation and chiral reflection will be important:

$$U(M_{ch}) = W(M_r) \oplus \bar{W}(M_l), \qquad \bar{U}(M_{ch}) = W(M_l) \oplus \bar{W}(M_r) \tag{3.5}$$

The sections of the dual bundles $U^*(M_{ch})$ and $\bar{U}^*(M_{ch})$ are called *chiral forms* of type $(1,0)_{ch}$ and $(0,1)_{ch}$ respectively. The corresponding chiral and antichiral differentials are defined by:

$$d_{ch} = \partial^r + \bar{\partial}^l, \qquad \bar{d}_{ch} = \partial^l + \bar{\partial}^r; \qquad d = d_{ch} + \bar{d}_{ch} \tag{3.6}$$

The structure preserving coordinate transformations on a chiral manifold are those holomorphic transformations, which do not mix the two sets of coordinates $\{z_r^M\}$ and $\{z_l^M\}$. These transformations are called *chiral coordinate transformations*. Thus, a chiral manifold is the product of two diffeomorphic complex manifolds M_r and M_l, which are called the right and left manifold respectively. Since chiral coordinate transformations leave M_r and M_l separately invariant, it is convenient to imagine the two manifolds forming a 'double layer' with functions on the chiral manifold depending on arbitrary pairs of points p_r and p_l on M_r and M_l respectively.

3.2 Connections on a Chiral Manifold

A *connection* D in the tangent bundle of a chiral manifold is a $\mathbb{C}$-linear map satisfying:

$$\begin{aligned} &\mathrm{D}: W(M_i) \to V^*_{\mathbb{C}}(M_{ch}) \otimes W(M_i) \\ &\mathrm{D}(f\psi^i) = df \otimes \psi^i + f\mathrm{D}\psi^i; \quad \psi^i \in W(M_i),\ f \in C(M_{ch}),\ i \in \{r, l\} \end{aligned} \tag{3.7}$$

This definition is extended from $W(M_i)$ to the associated bundles $\bar{W}(M_i)$, $W^*(M_i)$ and $\bar{W}^*(M_i)$ by the requirements of reality and of compatibility with index contraction. The Leibnitz rule then allows further extension of the connection to arbitrary tensor products of the tangent and cotangent bundles.

A connection is completely determined by its *connection matrices* Θ_r and Θ_l which are matrices of 1-forms defined by the action of D on a basis of the tangent bundle

$$\mathrm{D}e^i_A = e^i_B \Theta_i^B{}_A, \qquad \mathrm{D}\omega_i^A = -\Theta_i^A{}_B \omega_i^B, \qquad \mathrm{D}e^i_{\dot{A}} = e^i_{\dot{B}} \bar{\Theta}_i^{\dot{B}}{}_{\dot{A}}, \qquad \mathrm{D}\omega_i^{\dot{A}} = -\bar{\Theta}_i^{\dot{A}}{}_{\dot{B}} \omega_i^{\dot{B}} \tag{3.8}$$

where $\{e^i_A\}_{A\in\{1\ldots n\}}$ denotes a basis of $W(M_i)$ and $\{\omega_i^A\}$ the basis of $W^*(M_i)$ dual to $\{e^i_A\}$; the corresponding bases $\{e^i_{\dot{A}}\}$ of $\bar{W}(M_i)$ and $\{\omega_i^{\dot{A}}\}$ of $\bar{W}^*(M_i)$ are obtained by complex conjugation of the former bases.

The connection matrices Θ_i transform inhomogeneously under basis transformations in the tangent bundle, $e^i_{A'} = e^i_B K_i^{-1B}{}_{A'}$:

$$\Theta_i^{A'}{}_{B'} = K_i^{A'}{}_C \Theta_i^C{}_D K_i^{-1D}{}_{B'} + K_i^{A'}{}_C d K_i^{-1C}{}_{B'} \tag{3.9}$$

The *curvature matrices* Ω_i of the connection are defined by:

$$\Omega_i^A{}_B = d\Theta_i^A{}_B + \Theta_i^A{}_C \wedge \Theta_i^C{}_B \tag{3.10}$$

They are matrices of 2-forms which transform homogeneously under basis transformations. Exterior derivation of (3.10) yields the *first Bianchi identities:*

$$d\Omega_i^A{}_B + \Theta_i^A{}_C \wedge \Omega_i^C{}_B - \Omega_i^A{}_C \wedge \Theta_i^C{}_B = 0 \tag{3.11}$$

The right and left *torsion forms* τ_i of the connection are defined as follows:

$$\tau_i^A = d\omega_i^A + \Theta_i^A{}_B \wedge \omega_i^B \tag{3.12}$$

They are vectors of 2-forms, which transform homogeneously under basis transformations. Exterior derivation of (3.12) yields the *second Bianchi identities:*

$$d\tau_i^A + \Theta_i^A{}_B \wedge \tau_i^B = \Omega_i^A{}_B \wedge \omega_i^B \tag{3.13}$$

3.3 Dirac Connection

A *Dirac manifold* is a chiral manifold equipped with a symmetric bilinear product in its tangent bundle, which takes non-zero values only between sections of $W(M_r)$ and $\bar{W}(M_l)$ as well as between sections of $W(M_l)$ and $\bar{W}(M_r)$:

$$\begin{gathered} \begin{pmatrix} (e^r_{\dot{A}}, e^r_B) & (e^r_{\dot{A}}, e^l_B) \\ (e^l_{\dot{A}}, e^r_B) & (e^l_{\dot{A}}, e^l_B) \end{pmatrix} = \begin{pmatrix} 0 & g^+_{\dot{A}B} \\ g_{\dot{A}B} & 0 \end{pmatrix}, \\ (e^i_B, e^{i'}_{\dot{A}}) = (e^{i'}_{\dot{A}}, e^i_B), \qquad (e^i_A, e^{i'}_B) = (e^i_{\dot{A}}, e^{i'}_{\dot{B}}) = 0 \end{gathered} \tag{3.14}$$

The matrix $g_{\dot{A}B}$ is called the *Dirac metric* and $g^+_{\dot{A}B}$ denotes its hermitian adjoint. The Dirac metric gives rise to a 2-form of type $(1_r, 1_l)$ which is called the *chiral Kähler form*:

$$\hat{g} = i g_{\dot{M}N} dz_l^{\dot{M}} \wedge dz_r^N \tag{3.15}$$

The chiral Kähler form of a Dirac manifold is further required to satisfy the *chiral Kähler condition*, which forces the components of the metric to be analytic functions of z_r^M and $z_l^{\dot{M}}$:

$$\bar{d}_{ch}\hat{g} = 0 \quad \Leftrightarrow \quad \partial^l_S g_{\dot{M}N} = 0, \qquad \partial^r_{\dot{S}} g_{\dot{M}N} = 0 \tag{3.16}$$

Since ultimately only the equations on spacetime are of interest, analyticity is required only on this four-dimensional submanifold, allowing for singularities in other regions of the Dirac manifold.

The Dirac metric may be considered as a special case of a hermitian metric. As a consequence of the vanishing diagonal blocks, the corresponding canonical connection, which is called the *Dirac connection*, leaves the spaces $W(M_i)$ invariant, as required by the definition (3.7) of a connection on a chiral manifold. The connection matrices of the Dirac connection are given by

$$\Theta_r^M{}_N = g^{M\dot{S}}\partial g_{\dot{S}N}, \qquad \Theta_l^M{}_N = g^{+M\dot{S}}\partial g^+_{\dot{S}N} \tag{3.17}$$

where indices from the middle of the alphabet refer to a chiral coordinate basis. As a consequence of the chiral Kähler condition the differentials ∂^l and ∂^r annihilate the metric and its adjoint respectively, whence Θ_r and Θ_l are matrices of forms of type $(1_r, 0)$ and $(1_l, 0)$ respectively. Under chiral coordinate transformations the connection matrices retain their form (3.17), while in a general[8] basis they are given by

$$\Theta_i^A{}_B = E_i^A{}_M \Theta_i^M{}_N E_i^N{}_B + E_i^A{}_M dE_i^M{}_B \tag{3.18}$$

where $E_i^A{}_M$ denotes the transformation matrix from a coordinate basis to a general basis in $W(M_i)$ and $E_i^M{}_A$ its inverse.

The Dirac connection could alternatively be defined to be the unique connection satisfying the following two requirements:

D is compatible with the Dirac metric:

$$d(e^l_{\dot{A}}, e^r_B) = (\mathrm{D}e^l_{\dot{A}}, e^r_B) + (e^l_{\dot{A}}, \mathrm{D}e^r_B) \quad \Leftrightarrow \quad dg_{\dot{A}B} = g_{\dot{A}C}\Theta_r^C{}_B + \bar{\Theta}_l^{\dot{C}}{}_{\dot{A}} g_{\dot{C}B} \tag{3.19}$$

$\Theta_r^M{}_N$ and $\Theta_l^M{}_N$ are matrices of 1-forms of type $(1, 0)$ for chiral coordinate bases.

An important class of non-holomorphic bases are the *spinor bases* which are defined by the requirement that the Dirac metric be equal to the unit matrix:

$$\bar{E}_l^{\dot{M}}{}_{\dot{A}} g_{\dot{M}N} E_r^N{}_B = \delta_{\dot{A}B} \tag{3.20}$$

The transformation matrices E_r and E_l from a coordinate basis to a spinor basis are called right and left *dyad fields* (anticipating that n will be set equal to two in the next chapter), in analogy to the tetrad fields of general relativity. They are further required to satisfy the *chirality conditions*

$$\bar{d}_{ch}E_r^A{}_M = 0, \qquad d_{ch}E_l^A{}_M = 0 \tag{3.21}$$

which are compatible with the chiral Kähler condition on the Dirac metric. With this restriction on the dyad fields, comparison with (3.18) shows that Θ_r in a spinor basis is a matrix of chiral forms of type $(1, 0)_{ch}$, while the left connection matrix Θ_l consists of chiral forms

[8] A ‘general basis’ in the tangent bundle of a chiral manifold is not completely arbitrary since it is required to be compatible with the complex and chiral structures.

of type $(0,1)_{ch}$. The group $\mathcal{G}_n$ of basis transformations, which transform spinor bases into each other is isomorphic to $\mathbf{GL}(\mathbb{C}^n)$:

$$\mathcal{G}_n = \{(K_r, K_l) \in \mathbf{GL}(\mathbb{C}^n) \times \mathbf{GL}(\mathbb{C}^n) \mid K_l^+ K_r = \mathbf{1}_n\} \cong \mathbf{GL}(\mathbb{C}^n) \tag{3.22}$$

In a spinor basis the right and left connection matrices (with first index lowered with the Dirac metric and its adjoint respectively), are related by hermitian conjugation, as follows from the compatibility of the connection with the metric (3.19):

$$\Theta^l_{\dot{A}B} = -\Theta^{r+}_{\dot{A}B} \tag{3.23}$$

3.4 Curvature and Torsion of the Dirac Connection

The curvature matrices of the Dirac connection are matrices of 2-forms of type $(1,1)$, since the $(2,0)$-type terms on the right hand side of (3.10) vanish identically, as may be seen using a chiral coordinate basis:

$$\Omega_i^M{}_N = \bar{\partial}\Theta_i^M{}_N, \qquad \partial\Theta_i^M{}_N + \Theta_i^M{}_S \wedge \Theta_i^S{}_N = 0 \tag{3.24}$$

The chiral Kähler condition further restricts Ω_r and Ω_l to be forms of type $(1_r, 1_l)$ and $(1_l, 1_r)$ respectively. As a consequence of the compatibility of the connection with the metric the right and left curvature matrices are related by hermitian conjugation:

$$\Omega^r_{\dot{A}B} = -\Omega^{l+}_{\dot{A}B} \tag{3.25}$$

This may also be seen from the expressions of Ω_r and Ω_l in terms of the Dirac metric:

$$\Omega^r_{\dot{M}N} = \bar{\partial}\partial g_{\dot{M}N} - \bar{\partial} g_{\dot{M}R} \wedge g^{R\dot{S}}\partial g_{\dot{S}N}, \qquad \Omega^l_{\dot{M}N} = \bar{\partial}\partial g^+_{\dot{M}N} - \bar{\partial} g^+_{\dot{M}R} \wedge g^{+R\dot{S}}\partial g^+_{\dot{S}N} \tag{3.26}$$

An expansion of the curvature 2-forms in terms of a basis in the cotangent bundle defines the right and left *Riemann spinors:*

$$\Omega_r^A{}_B = R_r^A{}_{B\dot{C}D}\, \omega_l^{\dot{C}} \wedge \omega_r^D, \qquad \Omega_l^A{}_B = R_l^A{}_{B\dot{C}D}\, \omega_r^{\dot{C}} \wedge \omega_l^D \tag{3.27}$$

As a consequence of (3.25) they satisfy the following symmetry property:

$$R^r_{\dot{A}B\dot{C}D} = \bar{R}^l_{B\dot{A}D\dot{C}} \tag{3.28}$$

The right Riemann spinor may be contracted using the Dirac metric; one obtains four *Ricci spinors* and two *curvature scalars:*

$$\begin{aligned} R_{\dot{A}B} &= R_r^C{}_{B\dot{A}C}, & \tilde{R}_{\dot{A}B} &= R^r_{\dot{A}C}{}^C{}_B, & R &= R^B{}_B \\ \rho_{\dot{A}B} &= R_r^C{}_{C\dot{A}B}, & \tilde{\rho}_{\dot{A}B} &= R^r_{\dot{A}B}{}^C{}_C, & \rho &= \rho^B{}_B \end{aligned} \tag{3.29}$$

The index referring to the right manifold has been omitted from the Ricci spinors, since the corresponding left Ricci spinors, which are obtained by contracting the left Riemann spinor with the adjoint Dirac metric, may be expressed in terms of adjoints of the right Ricci spinors as a consequence of (3.28):

$$R_l^C{}_{B\dot{A}C} = \tilde{R}^+_{\dot{A}B}, \qquad R^l_{\dot{A}C}{}^C{}_B = R^+_{\dot{A}B}, \qquad R_l^C{}_{C\dot{A}B} = \rho^+_{\dot{A}B}, \qquad R^l_{\dot{A}B}{}^C{}_C = \tilde{\rho}^+_{\dot{A}B} \tag{3.30}$$

In a chiral coordinate basis the expressions (3.12) for the right and left torsion forms simplify

$$\tau_i^M = \Theta_i^M{}_N \wedge dz_i^N \tag{3.31}$$

and it is seen that τ_r and τ_l are vectors of forms of type $(2_r, 0)$ and $(2_l, 0)$ respectively. An expansion of these forms in terms of a basis in the cotangent bundle defines the right and left *torsion spinors:*

$$\tau_i^A = \frac{1}{2} T_i^A{}_{BC} \omega_i^B \wedge \omega_i^C \tag{3.32}$$

In analogy to the case of a hermitian manifold, on a *Dirac manifold of dimension* $n = 2$ there is a linear relation between the Ricci spinors

$$R_{\dot{A}B} + \tilde{R}_{\dot{A}B} - \rho_{\dot{A}B} - \tilde{\rho}_{\dot{A}B} + (\rho - R) g_{\dot{A}B} = 0 \tag{3.33}$$

and the torsion may be expressed in terms of the contracted torsion spinors:

$$T_i^A{}_{BC} = \delta_B^A T_C^i - \delta_C^A T_B^i, \qquad T_A^i = T_i^B{}_{BA} \tag{3.34}$$

4 Minkowski Bases on a Dirac Manifold

Spinor relativity is defined on a Dirac manifold of dimension two. In this case the tangent spaces carry the structures of Dirac spinor space and the sections of the tangent bundle in a spinor basis are called spinor fields. A distinguished spinor field, which arises naturally from the Dirac metric, is provided by the contracted torsion and is called the fundamental spinor field. Besides its significance as a physical field, the fundamental spinor field also enables the introduction of Minkowski bases. The sections of the tangent bundle in a Minkowski basis are called vector fields, since they are acted on by Lorentz transformations in their defining representation. In this chapter Minkowski bases are introduced and the structures of the Dirac manifold, in particular the Dirac connection, are expressed in terms of them.

4.1 Dirac Spinors

In case of a Dirac manifold of dimension $n = 2$ the invariance group $\mathcal{G}_2$ of the Dirac metric is isomorphic to a product of the covering group of the Lorentz group and two abelian factors; it is called the *gauge group* of spinor relativity:

$$\mathcal{G}_2 \cong \mathbf{GL}(\mathbb{C}^2) = \mathbf{D}(1) \times \mathbf{U}(1) \circ \mathbf{SL}(\mathbb{C}^2) \tag{4.1}$$

The sections of the tangent bundle in a spinor basis thus carry a spinor representation of the Lorentz group and will therefore be called *spinor fields*. A distinguished spinor field is present on a Dirac manifold in form of the contracted torsion spinors. Because of its importance in spinor relativity it is called the *fundamental spinor field* and the special symbols φ and χ are introduced for its right and left part respectively:

$$\varphi^A = i\delta^{A\dot{B}} \bar{T}_{\dot{B}}^l, \qquad \chi^A = -i\delta^{A\dot{B}} \bar{T}_{\dot{B}}^r \tag{4.2}$$

As a consequence of (3.21)[9] the fundamental spinor field satisfies the *chirality conditions:*

$$\bar{d}_{ch}\varphi^A = 0, \qquad d_{ch}\chi^A = 0 \tag{4.3}$$

One may introduce a (chiral) Dirac spinor notation, where the sections of $W(M_l) \oplus W(M_r)$ with respect to a spinor basis are written as four-component spinors with indices omitted. In this notation the gauge group action on spinor fields takes the form

$$\begin{aligned} &\psi \to K_{\mathcal{D}}\psi; \quad \psi = \begin{pmatrix} \phi_l \\ \phi_r \end{pmatrix}, \; K_{\mathcal{D}} = \begin{pmatrix} K_l & 0 \\ 0 & K_r \end{pmatrix} \\ &K_l = \exp\frac{1}{2}\{(i\vartheta_0 - \kappa_0) + (i\vec{\vartheta} - \vec{\kappa})\vec{\sigma}\}, \qquad K_r = K_l^{-1+} \end{aligned} \tag{4.4}$$

where $\vec{\sigma}$ denotes the Pauli matrices and the parameters κ_0, ϑ_0, $\vec{\kappa}$ and $\vec{\vartheta}$ are real. If ϑ_0 and κ_0 vanish, the upper and lower part of $K_{\mathcal{D}}$ reduce to the left and right Weyl-representation of $\mathbf{SL}(\mathbb{C}^2)$ respectively, where $\vec{\vartheta}$ generates rotations and $\vec{\kappa}$ generates boosts. The coordinate dependence of the parameters is restricted by the chirality conditions (3.21) on the dyad fields, which force the allowed gauge transformations to satisfy the corresponding conditions $\bar{d}_{ch}K_r = 0$ and $d_{ch}K_l = 0$.

The product in the tangent bundle of a Dirac manifold induces an indefinite hermitian product on $W(M_l) \oplus W(M_r)$, which in Dirac spinor notation takes the form:

$$\psi^+\beta\psi = \bar{\phi}_r^{\dot{A}}\delta_{\dot{A}B}\phi_l^B + \bar{\phi}_l^{\dot{A}}\delta_{\dot{A}B}\phi_r^B, \quad \beta = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} \tag{4.5}$$

Further, the natural projectors P_l and P_r onto the subbundles of $W(M_l) \oplus W(M_r)$ are given by:

$$P_l = \frac{1}{2}(\mathbf{1} + \gamma_5), \quad P_r = \frac{1}{2}(\mathbf{1} - \gamma_5); \quad \gamma_5 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \tag{4.6}$$

The gauge group of spinor relativity is isomorphic to the group of linear transformations of Dirac spinors which leave the hermitian product invariant and commute with the chiral projectors:

$$\mathcal{G}_2 \cong \{K_{\mathcal{D}} \in \mathbf{GL}(\mathbb{C}^4) \mid K_{\mathcal{D}}^+\beta K_{\mathcal{D}} = \beta\,,\ [K_{\mathcal{D}}, \gamma_5] = 0\,\} \tag{4.7}$$

4.2 Dirac and Weyl Matrices

Dirac matrices represent the Clifford algebra of the Lorentz group in Dirac spinor automorphisms:

$$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}; \qquad \eta^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1}_3 \end{pmatrix} \tag{4.8}$$

[9] This may be seen from the explicit expression of the torsion spinors in terms of the dyad fields

$$T_i{}^A{}_{BC} = E_i^A{}_M \partial_C^i E_i^M{}_B + \delta^{A\dot{E}}\bar{E}_{\tilde{i}}^{\dot{D}}{}_{\dot{M}}\partial_C^i \bar{E}_{\tilde{i}}^{\dot{M}}{}_{\dot{E}}\delta_{\dot{D}B} - E_i^A{}_M \partial_B^i E_i^M{}_C - \delta^{A\dot{E}}\bar{E}_{\tilde{i}}^{\dot{D}}{}_{\dot{M}}\partial_B^i \bar{E}_{\tilde{i}}^{\dot{M}}{}_{\dot{E}}\delta_{\dot{D}C}$$

where $\tilde{i}$ is the chirality index opposite to i and ∂_A^i denotes partial derivation in the direction of e_A^i.

In a chiral basis they take the form

$$\gamma^\alpha = -i \begin{pmatrix} 0 & \sigma^\alpha \\ -\hat{\sigma}^\alpha & 0 \end{pmatrix}, \quad \sigma^\alpha = (\mathbf{1}_2, \vec{\sigma}), \ \hat{\sigma}^\alpha = (-\mathbf{1}_2, \vec{\sigma}) \tag{4.9}$$

where σ^α and $\hat{\sigma}^\alpha$ are the right and left Weyl matrices respectively. In terms of Weyl matrices the anticommutation relations of Dirac matrices read:

$$\sigma_{(\alpha}\hat{\sigma}_{\beta)} = \hat{\sigma}_{(\alpha}\sigma_{\beta)} = \eta_{\alpha\beta} \tag{4.10}$$

The action of the gauge group gives rise to real Lorentz transformations of the Dirac matrices together with $\mathbf{D}(1)$-transformations involving γ_5

$$K_{\mathcal{D}}^{-1}\gamma^\alpha K_{\mathcal{D}} = \mathrm{e}^{\kappa_0\gamma_5}\Lambda^\alpha{}_\beta\gamma^\beta \tag{4.11}$$

where $K_{\mathcal{D}}$ is given by (4.4) and $\Lambda^\alpha{}_\beta$ is the Lorentz transformation generated by the antisymmetric matrix $\omega_{\alpha\beta}$ with components:

$$\omega_{k0} = \kappa_k, \qquad \frac{1}{2}\epsilon_{kmn}\omega_{mn} = \vartheta_k \tag{4.12}$$

This transformation property of the Dirac matrices may equivalently be expressed in terms of Weyl matrices:

$$K_l^{-1}\sigma^\alpha K_r = \mathrm{e}^{\kappa_0}\Lambda^\alpha{}_\beta\sigma^\beta, \qquad K_r^{-1}\hat{\sigma}^\alpha K_l = \mathrm{e}^{-\kappa_0}\Lambda^\alpha{}_\beta\hat{\sigma}^\beta \tag{4.13}$$

Using Dirac matrices the gauge transformation matrices (4.4) may be written in the form:

$$K_{\mathcal{D}} = \exp\left\{\frac{i}{2}\vartheta_0\mathbf{1}_4 - \frac{1}{2}\kappa_0\gamma_5 + \frac{1}{4}\omega_{\alpha\beta}S^{\alpha\beta}\right\}, \qquad S_{\alpha\beta} = \frac{1}{2}[\gamma_\alpha, \gamma_\beta] \tag{4.14}$$

The Lorentz generators S satisfy the following (anti-)commutation relations with Dirac matrices

$$\frac{1}{2}[\gamma_\alpha, S_{\beta\gamma}] = \eta_{\alpha\beta}\gamma_\gamma - \eta_{\alpha\gamma}\gamma_\beta, \qquad \frac{1}{2}\{\gamma_\alpha, S_{\beta\gamma}\} = i\epsilon_{\alpha\beta\gamma\delta}\gamma_5\gamma^\delta \tag{4.15}$$

where the sign convention $\epsilon_{0123} = -1$ for the totally antisymmetric tensor has been chosen. They are further selfdual in the following sense:

$$S^*_{\alpha\beta} = \frac{1}{2}S_{\gamma\delta}\epsilon^{\gamma\delta}{}_{\alpha\beta} = i\gamma_5 S_{\alpha\beta} \tag{4.16}$$

The Weyl matrices satisfy the Fiertz identities

$$\begin{aligned} \frac{1}{2}(\sigma^\alpha)^A{}_B(\hat{\sigma}_\alpha)^C{}_D &= \delta^A{}_D\delta^C{}_B \\ \frac{1}{2}(\sigma^\alpha)^A{}_B(\sigma_\alpha)^C{}_D &= \epsilon_l^{AC}\epsilon^r_{DB} \\ \frac{1}{2}(\hat{\sigma}^\alpha)^A{}_B(\hat{\sigma}_\alpha)^C{}_D &= \epsilon_r^{AC}\epsilon^l_{DB} \end{aligned} \tag{4.17}$$

and they are chirally reflected to each other as follows

$$\epsilon^l_{AC}(\sigma_\alpha)^C{}_D\epsilon_r^{DB} = (\hat{\sigma}_\alpha)^B{}_A, \qquad \epsilon^r_{AC}(\hat{\sigma}_\alpha)^C{}_D\epsilon_l^{DB} = (\sigma_\alpha)^B{}_A \tag{4.18}$$

where ϵ^i_{AB} denotes the antisymmetric matrices with $\epsilon^r_{12} = \epsilon^l_{21} = 1$ and ϵ_i^{AB} their inverses.

4.3 Spacetime Pauli Matrices

From (4.13) it is seen that products of right and left Weyl matrices are $\mathbf{D}(1)$-invariant. The antisymmetric part of these products may be expanded in terms of Pauli matrices as follows:

$$\sigma_{[\alpha}\hat{\sigma}_{\beta]} = \sigma_k \bar{\Sigma}^k_{\alpha\beta}, \qquad \hat{\sigma}_{[\alpha}\sigma_{\beta]} = -\sigma_k \Sigma^k_{\alpha\beta} \tag{4.19}$$

The matrices $\Sigma^k_{\alpha\beta}$ introduced on the right hand sides of these relations are called *spacetime Pauli matrices*; they are antisymmetric in their Lorentz indices with components:

$$\Sigma^k_{l0} = \delta_{kl}, \qquad \Sigma^k_{mn} = -i\epsilon_{kmn} \tag{4.20}$$

Their name is justified by their (anti-)commutation relations

$$\{\Sigma^m, \Sigma^n\}_{\alpha\beta} = 2\delta_{mn}\eta_{\alpha\beta}, \qquad [\Sigma^m, \Sigma^n]_{\alpha\beta} = 2i\epsilon_{kmn}\Sigma^k_{\alpha\beta}, \qquad [\Sigma^m, \bar{\Sigma}^n] = 0 \tag{4.21}$$

which show that the spacetime Pauli matrices and their complex conjugates form two commuting sets of generators of the rotation group. Further, the spacetime Pauli matrices are antiselfdual and their products with complex conjugate spacetime Pauli matrices are symmetric and trace-free:

$$\Sigma^{k*}_{\alpha\beta} = -i\Sigma^k_{\alpha\beta}; \qquad (\Sigma^m\bar{\Sigma}^n)_{[\alpha\beta]} = 0, \qquad (\Sigma^m\bar{\Sigma}^n)^\alpha{}_\alpha = 0 \tag{4.22}$$

Since the spacetime Pauli matrices together with their complex conjugates form a basis for antisymmetric tensors, the following identity may be derived using (4.21) and (4.22):

$$\frac{1}{2}\Sigma^{\delta\gamma}_k \Sigma^k_{\alpha\beta} = \eta^\gamma{}_{[\alpha}\eta^\delta{}_{\beta]} + \frac{i}{2}\epsilon^{\gamma\delta}{}_{\alpha\beta} \tag{4.23}$$

In products with Weyl matrices it is possible to switch between the two types of Pauli matrices using the relations:

$$\Sigma^\alpha_{k\ \beta}\sigma^\beta = \sigma^\alpha\sigma_k, \qquad \Sigma^\alpha_{k\ \beta}\hat{\sigma}^\beta = -\sigma_k\hat{\sigma}^\alpha \tag{4.24}$$

The Lorentz generators may be expressed in terms of the two types of Pauli matrices as follows:

$$S_{\alpha\beta} = \begin{pmatrix} \sigma_k\bar{\Sigma}^k_{\alpha\beta} & 0 \\ 0 & -\sigma_k\Sigma^k_{\alpha\beta} \end{pmatrix} \tag{4.25}$$

4.4 Spinor Tetrad Fields

From the fundamental spinor field (4.2) and Weyl matrices the *spinor tetrad fields* ξ and ζ are defined as follows

$$\xi^A_\alpha = \frac{(\hat{\sigma}_\alpha\chi)^A}{\sqrt{\varphi^+\chi}}, \qquad \zeta^A_\alpha = \frac{(\sigma_\alpha\varphi)^A}{\sqrt{\chi^+\varphi}}; \qquad \xi^\alpha_A = \eta^{\alpha\beta}\zeta^{\dot{B}}_\beta\delta_{\dot{A}B}, \qquad \zeta^\alpha_A = \eta^{\alpha\beta}\xi^{\dot{B}}_\beta\delta_{\dot{A}B} \tag{4.26}$$

where a dotted spinor index is used to denote the complex conjugate spinor tetrad fields. They carry a spinor and a vector index and will be used in the next section to associate a

Minkowski basis to each spinor basis. Under a gauge transformation the spinor tetrad fields transform according to the relations

$$\begin{aligned} \xi^A_\alpha &\to \mathrm{e}^{-\kappa_0} K_r{}^A{}_B \xi^B_\beta \Lambda^{-1\beta}{}_\alpha, \qquad \xi^\alpha_A \to \mathrm{e}^{\kappa_0} \Lambda^\alpha{}_\beta \xi^\beta_B K_r^{-1B}{}_A \\ \zeta^A_\alpha &\to \mathrm{e}^{\kappa_0} K_l{}^A{}_B \zeta^B_\beta \Lambda^{-1\beta}{}_\alpha, \qquad \zeta^\alpha_A \to \mathrm{e}^{-\kappa_0} \Lambda^\alpha{}_\beta \zeta^\beta_B K_l^{-1B}{}_A \end{aligned} \tag{4.27}$$

as may be seen from (4.13). This shows that the spinor tetrad fields are well suited for transforming spinor fields, acted on by spinor representations of the Lorentz group, to vector fields, acted on by real Lorentz transformations.

Contracting Lorentz indices in products of spinor tetrad fields yields the identities

$$\frac{1}{2}\xi^A_\gamma \xi^\gamma_B = \delta^A{}_B, \qquad \frac{1}{2}\zeta^A_\gamma \zeta^\gamma_B = \delta^A{}_B; \qquad \xi^A_\gamma \zeta^\gamma_{\dot B} = 0, \qquad \zeta^A_\gamma \xi^\gamma_{\dot B} = 0 \tag{4.28}$$

as follows from (4.17). Contracting spinor indices in products of spinor tetrad fields one obtains tensor fields; from the symmetric product the Minkowski metric is recovered, while the antisymmetric product defines a new $\mathbf{D}(1)$-invariant tensor field I, which is called the *complex spinor structure:*

$$\delta_{\dot A B}\zeta^{\dot A}_{(\alpha}\xi^B_{\beta)} = \eta_{\alpha\beta}, \qquad i\delta_{\dot A B}\zeta^{\dot A}_{[\alpha}\xi^B_{\beta]} = i n_k \bar\Sigma^k_{\alpha\beta} = I_{\alpha\beta}; \qquad n^k = \frac{\varphi^+\sigma_k\chi}{\varphi^+\chi}, \qquad \vec n^2 = 1 \tag{4.29}$$

As indicated by this name, the complex structure in the tangent bundle of a Dirac manifold is represented on vector fields by I. This will be seen in the next section to be a consequence of the identities

$$I^\alpha{}_\beta \xi^\beta_C = i\xi^\alpha_C, \qquad I^\alpha{}_\beta \zeta^\beta_{\dot C} = -i\zeta^\alpha_{\dot C}; \qquad \xi^C_\beta I^\beta{}_\alpha = i\xi^C_\alpha, \qquad \zeta^{\dot C}_\beta I^\beta{}_\alpha = -i\zeta^{\dot C}_\alpha \tag{4.30}$$

which are derived with help of (4.28). I has the following properties

$$(I^2)^\alpha{}_\beta = -\eta^\alpha{}_\beta, \qquad I^*_{\alpha\beta} = iI_{\alpha\beta}, \qquad [I, \Sigma^k]_{\alpha\beta} = 0, \qquad \{I, \bar\Sigma^k\}_{\alpha\beta} = 2in_k\eta_{\alpha\beta} \tag{4.31}$$

the first of which makes it a candidate for a complex structure. It is further possible to express the totally antisymmetric tensor in terms of the complex spinor structure

$$\frac{i}{2}\epsilon^{\gamma\delta}{}_{\alpha\beta} = I^\gamma{}_{[\alpha} I^\delta{}_{\beta]} - \frac{1}{2}I^{\gamma\delta} I_{\alpha\beta} \tag{4.32}$$

as may be verified by contracting (4.32) with Σ^k and $\bar\Sigma^k$ respectively, remembering that these six matrices constitute a basis for antisymmetric tensors.

4.5 Minkowski Bases

The chiral coordinate and spinor bases in the tangent bundle of a Dirac manifold considered up to now were compatible with the complex and chiral structures, i.e. the transformation matrices relating them commute with I and J and thus leave the bundles $W(M_i)$ and their associated bundles invariant. This is no longer the case for *Minkowski bases*, which combine basis vectors from bundles complex conjugate and chirally reflected to each other.

For a given spinor basis the associated Minkowski basis $\{\mathcal{E}_\alpha\}_{\alpha\in\{0\dots3\}}$ of $U(M_{ch})$ and the corresponding dual basis $\{\vartheta^\alpha\}_{\alpha\in\{0\dots3\}}$ of $U^*(M_{ch})$ are defined as follows:

$$\mathcal{E}_\alpha = \frac{1}{2}(\xi_\alpha^A e_A^r + \zeta_\alpha^{\dot A} e_{\dot A}^l), \qquad \vartheta^\alpha = \xi_A^\alpha \omega_r^A + \zeta_{\dot A}^\alpha \omega_l^{\dot A} \tag{4.33}$$

On the complex conjugate bundles $\bar U(M_{ch})$ and $\bar U^*(M_{ch})$ the complex conjugate Minkowski bases $\{\bar{\mathcal{E}}_\alpha\}$ and $\{\bar\vartheta^\alpha\}$ respectively are introduced. The inverse basis transformations are obtained with help of the identities (4.28):

$$e_A^r = \xi_A^\alpha \mathcal{E}_\alpha, \qquad e_A^l = \zeta_A^\alpha \bar{\mathcal{E}}_\alpha; \qquad \omega_r^A = \frac{1}{2}\xi_\alpha^A \vartheta^\alpha, \qquad \omega_l^A = \frac{1}{2}\zeta_\alpha^A \bar\vartheta^\alpha \tag{4.34}$$

The sections of the tangent bundle in a Minkowski basis are called (complex) *vector fields*. The product in the tangent bundle of a Dirac manifold takes the form of a symmetric bilinear product between vector fields given by the Minkowski metric:

$$(\mathcal{E}_\alpha, \mathcal{E}_\beta) = (\bar{\mathcal{E}}_\alpha, \bar{\mathcal{E}}_\beta) = \frac{1}{2}\eta_{\alpha\beta}, \qquad (\mathcal{E}_\alpha, \bar{\mathcal{E}}_\beta) = (\bar{\mathcal{E}}_\alpha, \mathcal{E}_\beta) = 0 \tag{4.35}$$

The factor in front of the Minkowski metric is a consequence of the normalization of the basis vectors, which has been chosen such that inconvenient factors of $\sqrt{2}$ are avoided later on. In the following tensor indices are always raised and lowered with the Minkowski metric without extra factors, as had already been tacitly assumed in the previous sections.

The action of gauge transformations on Minkowski bases is obtained with help of (4.27)

$$\begin{aligned} &\omega_r^A \to K_r{}^A{}_B \omega_r^B, \qquad \omega_l^A \to K_l{}^A{}_B \omega_l^B \quad \Rightarrow \\ &\vartheta^\alpha \to \Lambda^\alpha{}_\beta[e^{\kappa_0}\xi_A^\alpha \omega_r^A + e^{-\kappa_0}\zeta_{\dot A}^\alpha \omega_l^{\dot A}] = \Lambda^\alpha{}_\beta[\mathrm{ch}\,\kappa_0 \eta^\beta{}_\gamma - i\,\mathrm{sh}\,\kappa_0 I^\beta{}_\gamma]\vartheta^\gamma = (\Lambda\mathcal{I})^\alpha{}_\beta \vartheta^\beta \end{aligned} \tag{4.36}$$

where (4.30) has been used in the second step. This shows that $\mathbf{SL}(\mathbb{C}^2)$-transformations are represented by real Lorentz transformations and $\mathbf{U}(1)$-transformations are represented trivially, as required for a vector basis. $\mathbf{D}(1)$-transformations are represented by complex matrices $\mathcal{I}$ involving the complex spinor structure. The action of the complex and chiral structures is given on Minkowski bases by I and $-iI$ respectively, as may be verified with help of (4.30):

$$\begin{aligned} &I: \omega_r^A \to i\omega_r^A, \qquad \omega_l^A \to i\omega_l^A \quad \Rightarrow \quad \vartheta^\alpha \to i[\xi_A^\alpha \omega_r^A - \zeta_{\dot A}^\alpha \omega_l^{\dot A}] = I^\alpha{}_\beta \vartheta^\beta \\ &J: \omega_r^A \to \omega_r^A, \qquad \omega_l^A \to -\omega_l^A \quad \Rightarrow \quad \vartheta^\alpha \to \xi_A^\alpha \omega_r^A - \zeta_{\dot A}^\alpha \omega_l^{\dot A} = -iI^\alpha{}_\beta \vartheta^\beta \end{aligned} \tag{4.37}$$

This shows that Minkowski bases are compatible with the new complex structure $\widetilde{I} = IJ$, which is called the *complex vector structure* and is represented on vectors by i.

Using Minkowski bases, the chiral differential (3.6) takes a particularly simple form involving the derivatives ∂_α^{ch} in the directions of the basis vectors $\mathcal{E}_\alpha$

$$d_{ch} = a^+(\vartheta^\alpha)\partial_\alpha^{ch}, \qquad \partial_\alpha^{ch} = \frac{1}{2}(\xi_\alpha^M \partial_M^r + \zeta_\alpha^{\dot M}\partial_{\dot M}^l) \tag{4.38}$$

where the tetrad fields with coordinate spinor index are given by:

$$\xi_\alpha^M = E_r{}^M{}_A \xi_\alpha^A, \qquad \zeta_\alpha^M = E_l{}^M{}_A \zeta_\alpha^A; \qquad \xi_M^\alpha = \xi_A^\alpha E_r{}^A{}_M, \qquad \zeta_M^\alpha = \zeta_A^\alpha E_l{}^A{}_M \tag{4.39}$$

Exterior derivation of the basis forms yields the structure equation

$$d\vartheta^\alpha = -\frac{1}{2}(\mathcal{C}^\alpha{}_{\beta\gamma}\vartheta^\beta \wedge \vartheta^\gamma + \widetilde{\mathcal{C}}^\alpha{}_{\beta\gamma}\vartheta^\beta \wedge \bar{\vartheta}^\gamma) \tag{4.40}$$

with complex structure functions:

$$\mathcal{C}^\alpha{}_{\beta\gamma} = \xi^M_\beta \partial^{ch}_\gamma \xi^\alpha_M + \zeta^{\dot{M}}_\beta \partial^{ch}_\gamma \zeta^\alpha_{\dot{M}}, \qquad \widetilde{\mathcal{C}}^\alpha{}_{\beta\gamma} = \xi^M_\beta \bar{\partial}^{ch}_\gamma \xi^\alpha_M + \zeta^{\dot{M}}_\beta \bar{\partial}^{ch}_\gamma \zeta^\alpha_{\dot{M}} \tag{4.41}$$

As a consequence of the chirality conditions (3.21) the antichiral derivatives in the expression for $\widetilde{\mathcal{C}}$ do not act on the dyad fields within the tetrad fields (4.39) and the expression may be evaluated further as follows:

$$\widetilde{\mathcal{C}}^\alpha{}_{\beta\gamma} = \xi^C_\beta \bar{\partial}^{ch}_\gamma \xi^\alpha_C + \zeta^{\dot{C}}_\beta \bar{\partial}^{ch}_\gamma \zeta^\alpha_{\dot{C}} = (\varphi^+\chi)^{-1}[(\bar{\partial}^{ch}_\gamma \varphi^+)\sigma_k \chi - \varphi^+\sigma_k \bar{\partial}^{ch}_\gamma \chi]\,\bar{\Sigma}^\alpha_{k\,\beta} \tag{4.42}$$

In the second step the relations (4.10) and (4.19) for products of Weyl matrices have been used as well as (4.29), which may be written more conveniently in the form:

$$\frac{1}{2}(\xi^\alpha_C \xi^C_\beta + \zeta^\alpha_{\dot{C}} \zeta^{\dot{C}}_\beta) = \eta^\alpha{}_\beta, \qquad \frac{i}{2}(\xi^\alpha_C \xi^C_\beta - \zeta^\alpha_{\dot{C}} \zeta^{\dot{C}}_\beta) = I^\alpha{}_\beta \tag{4.43}$$

(4.42) shows in particular that $\widetilde{\mathcal{C}}$ is antisymmetric in its first index pair. The structure functions $\mathcal{C}$ on the other hand may be expressed in terms of the structure functions of the spinor basis.[10]

4.6 Dirac Connection in a Minkowski Basis

The action of the Dirac connection on Minkowski bases

$$\mathrm{D}\mathcal{E}_\alpha = \mathcal{E}_\beta \Theta^\beta{}_\alpha, \qquad \mathrm{D}\vartheta^\alpha = -\Theta^\alpha{}_\beta \vartheta^\beta, \qquad \mathrm{D}\bar{\mathcal{E}}_\alpha = \bar{\mathcal{E}}_\beta \bar{\Theta}^\beta{}_\alpha, \qquad \mathrm{D}\bar{\vartheta}^\alpha = -\bar{\Theta}^\alpha{}_\beta \bar{\vartheta}^\beta \tag{4.44}$$

is determined by the connection matrix $\Theta^\alpha{}_\beta$, which is derived from the general transformation behaviour of connection matrices applied to the transformation (4.33) on $U(M_{ch})$:

$$2\Theta^\alpha{}_\beta = \xi^\alpha_A \Theta^A_r{}_B \xi^B_\beta + \zeta^\alpha_{\dot{A}} \bar{\Theta}^{\dot{A}}_l{}_{\dot{B}} \zeta^{\dot{B}}_\beta + \xi^\alpha_C d\xi^C_\beta + \zeta^\alpha_{\dot{C}} d\zeta^{\dot{C}}_\beta \tag{4.45}$$

This is equivalent to the requirement that the spinor tetrad fields be covariantly constant:

$$0 = \mathrm{D}\xi^\alpha_A = d\xi^\alpha_A + \Theta^\alpha{}_\beta \xi^\beta_A - \xi^\alpha_B \Theta^B_r{}_A, \qquad 0 = \mathrm{D}\zeta^\alpha_{\dot{A}} = d\zeta^\alpha_{\dot{A}} + \Theta^\alpha{}_\beta \zeta^\beta_{\dot{A}} - \zeta^\alpha_{\dot{B}} \bar{\Theta}^{\dot{B}}_l{}_{\dot{A}} \tag{4.46}$$

In order to simplify the expression (4.45), the connection matrices in a spinor basis are expanded in terms of Pauli matrices and the unit matrix using the symmetry property (3.23)

$$\Theta^A_r{}_B = -i\mathcal{A}\delta^A{}_B + L^k \sigma^A_k{}_B, \qquad \Theta^A_l{}_B = -i\bar{\mathcal{A}}\delta^A{}_B - \bar{L}^k \sigma^A_k{}_B \tag{4.47}$$

[10] With spinor structure functions defined by

$$d\omega^A_i + c^A_i{}_{BC}\omega^B_i \wedge \omega^C_i + c^A_i{}_{B\dot{C}}\omega^B_i \wedge \omega^{\dot{C}}_{\tilde{i}} = 0; \qquad c^A_i{}_{BC} = E^M_i{}_B \partial^i_C E^A_i{}_M, \qquad c^A_i{}_{B\dot{C}} = E^M_i{}_B \partial^{\tilde{i}}_{\dot{C}} E^A_i{}_M$$

where $\tilde{i}$ denotes the chirality index opposite to i, the Minkowski structure functions $\mathcal{C}$ take the form:

$$2\mathcal{C}^\alpha{}_{\beta\gamma} = \xi^\alpha_A[c^A_r{}_{BC}\xi^C_\gamma + c^A_r{}_{B\dot{C}}\zeta^{\dot{C}}_\gamma]\xi^B_\beta + \zeta^\alpha_{\dot{A}}[\bar{c}^{\dot{A}}_l{}_{\dot{B}\dot{C}}\zeta^{\dot{C}}_\gamma + \bar{c}^{\dot{A}}_l{}_{\dot{B}C}\xi^C_\gamma]\zeta^{\dot{B}}_\beta.$$

where $\mathcal{A}$ and L^k are chiral 1-forms of type $(1,0)_{ch}$. $\mathcal{A}$ is the *complex electromagnetic potential*, since it represents the $\mathbf{U}(1)$ and $\mathbf{D}(1)$ parts of the connection acting on spinor fields. These expansions are inserted into (4.45), the Pauli matrices are replaced with spacetime Pauli matrices using (4.24) and the resulting expression is simplified with help of (4.43). In the last two terms of (4.45) the total differentials may further be replaced with antichiral differentials as a consequence of the chirality conditions (4.3). The connection matrices in a Minkowski basis finally take the form:

$$\Theta^\alpha{}_\beta = -\mathcal{A} I^\alpha{}_\beta + L^k \Sigma^\alpha_{k\,\beta} + \frac{1}{2}(\xi^\alpha_C \bar{d}_{ch} \xi^C_\beta + \zeta^\alpha_{\dot{C}} \bar{d}_{ch} \zeta^{\dot{C}}_\beta) \tag{4.48}$$

Since $\mathcal{A}$ and L^k are chiral forms of type $(1,0)_{ch}$, their expansion in terms of Minkowski basis forms does not contain complex conjugate forms:

$$\mathcal{A} = \mathcal{A}_\alpha \vartheta^\alpha, \qquad L^k = L^k_\alpha \vartheta^\alpha \tag{4.49}$$

The last two terms of (4.48) on the other hand consist of chiral forms of type $(0,1)_{ch}$ and further comparison with (4.41) shows that they may be expressed in terms of the structure functions $\widetilde{\mathcal{C}}$. The connection matrix may thus be expanded as follows:

$$\Theta^\alpha{}_\beta = \Delta^\alpha{}_{\beta\gamma} \vartheta^\gamma - \frac{1}{2}\widetilde{\mathcal{C}}^\alpha{}_{\beta\gamma} \bar{\vartheta}^\gamma, \qquad \Delta^\alpha{}_{\beta\gamma} = -\mathcal{A}_\gamma I^\alpha{}_\beta + L^k_\gamma \Sigma^\alpha_{k\,\beta} \tag{4.50}$$

$\Theta_{\alpha\beta}$ is antisymmetric and consequently the Minkowski metric on $U(M_{ch})$ is covariantly constant, as required by the compatibility of the connection with the Dirac metric. The compatibility of the connection with the complex structure further requires I to be covariantly constant, which follows from (4.43) and the covariant constancy of the spinor tetrad fields.[11]

4.7 Curvature and Torsion in a Minkowski Basis

The curvature matrices in a Minkowski basis are obtained from the homogeneous transformation behaviour of the curvature:[12]

$$\Omega^\alpha{}_\beta = d\Theta^\alpha{}_\beta + \Theta^\alpha{}_\gamma \wedge \Theta^\gamma{}_\beta = \frac{1}{2}(\xi^\alpha_A \Omega^A_r{}_B \xi^B_\beta + \zeta^\alpha_{\dot{A}} \bar{\Omega}^{\dot{A}}_l{}_{\dot{B}} \zeta^{\dot{B}}_\beta) \tag{4.51}$$

[11] One may also verify this explicitly. Since I commutes with Σ^k, only the $\tilde{\mathcal{C}}$-part of the connection matrix enters the covariant derivative of I

$$(\mathrm{D}I)^\alpha{}_\beta = dI^\alpha{}_\beta + \Theta^\alpha{}_\gamma I^\gamma{}_\beta - I^\alpha{}_\gamma \Theta^\gamma{}_\beta = \bar{d}_{ch} I^\alpha{}_\beta + \frac{1}{2}(\varphi^+\chi)^{-1}[\varphi^+\sigma_k \bar{d}_{ch}\chi - (\bar{d}_{ch}\varphi^+)\sigma_k\chi][\bar{\Sigma}^k, I]^\alpha{}_\beta$$

where $\tilde{\mathcal{C}}$ has been expressed in its form (4.42) and the chirality condition $d_{ch} I^\alpha{}_\beta = 0$ has been used. This is evaluated further using (4.29) and the commutator of spacetime Pauli matrices

$$\begin{aligned}(\mathrm{D}I)^\alpha{}_\beta &= i(\varphi^+\chi)^{-1}[\bar{d}_{ch}(\varphi^+\sigma_k\chi) - n_k \bar{d}_{ch}(\varphi^+\chi)]\bar{\Sigma}^\alpha_{k\,\beta} \\ &\quad - (\varphi^+\chi)^{-1}\epsilon_{krs} n_r [\varphi^+\sigma_s \bar{d}_{ch}\chi - (\bar{d}_{ch}\varphi^+)\sigma_s\chi]\bar{\Sigma}^\alpha_{k\,\beta} = 0\end{aligned}$$

where the identity $i\epsilon_{kmn}\sigma^A_m{}_B \sigma^C_n{}_D = \delta^A{}_D \sigma^C_k{}_B - \sigma^A_k{}_D \delta^C{}_B$ for Pauli matrices has been used in the last step.

[12] The curvature matrix in a Minkowski basis may also be obtained from the exterior derivative of (4.48)

$$\begin{aligned}\Omega^\alpha{}_\beta &= d\Theta^\alpha{}_\beta + \Theta^\alpha{}_\gamma \wedge \Theta^\gamma{}_\beta \\ &= \left[\partial^{ch}_\gamma \Delta^\alpha{}_{\beta\delta} + \Delta^\alpha{}_{\epsilon\gamma}\Delta^\epsilon{}_{\beta\delta} - \frac{1}{2}\Delta^\alpha{}_{\beta\epsilon}\mathcal{C}^\epsilon{}_{\gamma\delta}\right]\vartheta^\gamma \wedge \vartheta^\delta + [dX^\alpha{}_\beta + X^\alpha{}_\gamma \wedge X^\gamma{}_\beta] \\ &\quad + \bar{d}_{ch}[-\mathcal{A} I^\alpha{}_\beta + L^k \Sigma^\alpha_{k\,\beta}] - \mathcal{A} \wedge [I, X]^\alpha{}_\beta + L^k \wedge [\Sigma_k, X]^\alpha{}_\beta\end{aligned}$$

In order to simplify this expression, the curvature matrices in a spinor basis are expanded in terms of Pauli matrices and the unit matrix in analogy to (4.47):

$$\Omega_r^A{}_B = -i\mathcal{F}\delta^A{}_B + Q^k\sigma_k^A{}_B, \qquad \Omega_l^A{}_B = -i\bar{\mathcal{F}}\delta^A{}_B - \bar{Q}^k\sigma_k^A{}_B \tag{4.52}$$

$\mathcal{F}$ is the *complex electromagnetic field*. Inserting these expansions into (4.51) and simplifying the resulting expression, the curvature matrix in a Minkowski basis takes the form:

$$\Omega^\alpha{}_\beta = -\mathcal{F}I^\alpha{}_\beta + Q^k\Sigma_k^\alpha{}_\beta \tag{4.53}$$

Since $\mathcal{F}$ and Q^k are chiral forms of type $(2,0)_{ch}$, they involve chiral differentials of the potential forms $\mathcal{A}$ and L^k, while the antichiral differentials of the latter vanish:

$$\mathcal{F} = d_{ch}\mathcal{A}, \qquad Q^k = d_{ch}L^k + i\epsilon_{kmn}L^m\wedge L^n; \qquad \bar{d}_{ch}\mathcal{A} = 0, \qquad \bar{d}_{ch}L^k = 0 \tag{4.54}$$

An expansion of $\mathcal{F}$ and Q^k in a Minkowski basis does not contain complex conjugate basis forms:

$$\mathcal{F} = \frac{1}{2}\mathcal{F}_{\alpha\beta}\vartheta^\alpha\wedge\vartheta^\beta, \qquad Q^k = \frac{1}{2}Q^k_{\alpha\beta}\vartheta^\alpha\wedge\vartheta^\beta \tag{4.55}$$

The corresponding expansion of the curvature matrix defines the *curvature tensor:*

$$\Omega^\alpha{}_\beta = \frac{1}{2}\mathcal{R}^\alpha{}_{\beta\gamma\delta}\vartheta^\gamma\wedge\vartheta^\delta, \qquad \mathcal{R}^\alpha{}_{\beta\gamma\delta} = -I^\alpha{}_\beta\mathcal{F}_{\gamma\delta} + \Sigma_k^\alpha{}_\beta Q^k_{\gamma\delta} \tag{4.56}$$

Since it is antisymmetric in both its first and last index pair, there is only one possibility (up to sign) to contract it, which yields the *Dirac-Ricci tensor:*

$$\mathcal{R}_{\alpha\beta} = \mathcal{R}^\gamma{}_{\alpha\gamma\beta}, \qquad \mathcal{R}_{\alpha\beta} = I_\alpha{}^\gamma\mathcal{F}_{\gamma\beta} - \Sigma_\alpha^{k\gamma}Q^k_{\gamma\beta} \tag{4.57}$$

The torsion form transforms homogeneously:

$$\tau^\alpha = d\vartheta^\alpha + \Theta^\alpha{}_\beta\wedge\vartheta^\beta = \xi_A^\alpha\tau_r^A + \zeta_{\dot{A}}^\alpha\bar{\tau}_l^{\dot{A}} \tag{4.58}$$

Using the relation (3.34) valid on Dirac manifolds of dimension two, the torsion forms in a spinor basis (3.32) may be expressed in terms of the fundamental spinor field (4.2) as follows:

$$\tau_r^A = -i\chi_B^+\omega_r^A\wedge\omega_r^B, \qquad \tau_l^A = i\varphi_B^+\omega_l^A\wedge\omega_l^B \tag{4.59}$$

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where $X^\alpha{}_\beta = -\frac{1}{2}\tilde{C}^\alpha{}_{\beta\gamma}\bar{\vartheta}^\gamma = \frac{1}{2}(\xi_C^\alpha\bar{d}_{ch}\xi_\beta^C + \zeta_{\dot{C}}^\alpha\bar{d}_{ch}\zeta_\beta^{\dot{C}})$. The second term in brackets vanishes:

$$d_{ch}X^\alpha{}_\beta = 0, \qquad \bar{d}_{ch}X^\alpha{}_\beta = -X^\alpha{}_\gamma\wedge X^\gamma{}_\beta = \frac{1}{2}(\bar{d}_{ch}\xi_C^\alpha\wedge\bar{d}_{ch}\xi_\beta^C + \bar{d}_{ch}\zeta_{\dot{C}}^\alpha\wedge\bar{d}_{ch}\zeta_\beta^{\dot{C}})$$

Using (4.54) the third and fourth term reduce to the covariant derivative of I, which vanishes:

$$\mathcal{A}\wedge(\bar{d}_{ch}I^\alpha{}_\beta + [X, I]^\alpha{}_\beta) = \mathcal{A}\wedge(\mathrm{D}I)^\alpha{}_\beta = 0$$

Finally, the fifth term vanishes since X commutes with Σ^k as a consequence of (4.42). The remaining first term yields the following expression for the curvature tensor:

$$\mathcal{R}^\alpha{}_{\beta\gamma\delta} = \partial_\gamma^{ch}\Delta^\alpha{}_{\beta\delta} - \partial_\delta^{ch}\Delta^\alpha{}_{\beta\gamma} - \Delta^\alpha{}_{\beta\epsilon}C^\epsilon{}_{[\gamma\delta]} + \Delta^\alpha{}_{\epsilon\gamma}\Delta^\epsilon{}_{\beta\delta} - \Delta^\alpha{}_{\epsilon\delta}\Delta^\epsilon{}_{\beta\gamma}.$$

Inserting these relations into (4.58) and expanding in a Minkowski basis yields:

$$\tau^\alpha = -\frac{i}{4}[\xi^\alpha_A \xi^A_\beta \, \chi^+_B \xi^B_\gamma + \zeta^\alpha_{\dot{A}} \zeta^{\dot{A}}_\beta \, \varphi_{\dot{B}} \zeta^{\dot{B}}_\gamma] \, \vartheta^\beta \wedge \vartheta^\gamma = \frac{1}{2} T^\alpha{}_{\beta\gamma} \, \vartheta^\beta \wedge \vartheta^\gamma \tag{4.60}$$

The *torsion tensor* thus defined may be simplified introducing the vector fields associated with the fundamental spinor field as follows:

$$T_\alpha = \frac{1}{2}(\varphi_{\dot{A}} \zeta^{\dot{A}}_\alpha + \chi^+_A \xi^A_\alpha), \qquad J_\alpha = \frac{1}{2}(\varphi_{\dot{A}} \zeta^{\dot{A}}_\alpha - \chi^+_A \xi^A_\alpha) \tag{4.61}$$

T_α and J_α are called the *torsion vector* and the *current vector* respectively. They are orthogonal to each other and have opposite squares:

$$T_\alpha T^\alpha = -J_\alpha J^\alpha = \chi^+ \varphi, \qquad T_\alpha J^\alpha = 0; \qquad T_\alpha = i J_\beta I^\beta{}_\alpha, \qquad J_\alpha = i T_\beta I^\beta{}_\alpha \tag{4.62}$$

Inserting these definitions into (4.60) and simplifying the resulting expression with help of (4.43), the torsion tensor may be written in the form:

$$T^\alpha{}_{\beta\gamma} = I^\alpha{}_{[\beta} J_{\gamma]} - i \eta^\alpha{}_{[\beta} T_{\gamma]} \tag{4.63}$$

From (4.58) and the structure equation (4.40) one obtains a relation between the torsion tensor, the connection coefficients (4.50) and the structure functions (4.41) as follows:

$$T^\alpha{}_{\beta\gamma} + \mathcal{C}^\alpha{}_{[\beta\gamma]} = \Delta^\alpha{}_{\gamma\beta} - \Delta^\alpha{}_{\beta\gamma} \tag{4.64}$$

Since the connection coefficients are antisymmetric in their first index pair, (4.64) may be solved for them:

$$\Delta_{\alpha\beta\gamma} = \frac{1}{2}(\mathcal{C}_{\gamma[\alpha\beta]} + \mathcal{C}_{\beta[\alpha\gamma]} - \mathcal{C}_{\alpha[\beta\gamma]}) + \mathcal{K}_{\alpha\beta\gamma}, \qquad \mathcal{K}_{\alpha\beta\gamma} = \frac{1}{2}[T_{\gamma\alpha\beta} + T_{\beta\alpha\gamma} - T_{\alpha\beta\gamma}] \tag{4.65}$$

The tensor $\mathcal{K}$ thus defined is called the *contortion tensor* and may be expressed in terms of the vector fields as follows:

$$\mathcal{K}_{\alpha\beta\gamma} = -\frac{1}{2}[I_{\alpha\beta} J_\gamma + i(\eta_{\alpha\gamma} T_\beta - \eta_{\beta\gamma} T_\alpha)] \tag{4.66}$$

5 Field Equations of Spinor Relativity

The dynamical equations of spinor relativity are obtained in analogy to general relativity from an action principle with action functional given by an integral of the curvature scalar R. The requirement that the action be stationary with respect to variations of the Dirac metric yields a differential equation, which is called the spinor Einstein equation and which relates the Ricci spinor $R_{\dot{A}B}$ to the fundamental spinor field. The equations of spinor relativity are then reexpressed in terms of a Minkowski basis, yielding tensor field equations. The spinor Einstein equation together with the second Bianchi identities takes the form of gravitational and electromagnetic Einstein equations, relating the Einstein tensor of the Dirac connection and the complex electromagnetic field tensor to functions of the fundamental spinor field. The first Bianchi identities yield equations for the covariant divergences of both the Einstein tensor and the dual electromagnetic field tensor.

5.1 Spinor Einstein Equation

As a consequence of the chiral Kähler condition $g_{\dot{M}N}$ may not be varied freely on the whole chiral manifold, but only on a four-dimensional submanifold $\mathcal{S}$. Such a manifold may be defined by the requirement that the right and left coordinates (for particular chiral coordinates) be equal:

$$z_r^M|_{\mathcal{S}} = z_l^M|_{\mathcal{S}} = z^M \tag{5.1}$$

The action of spinor relativity is given by an integral over $\mathcal{S}$ of the curvature scalar R

$$S = 2\mathrm{Re}\int_{\mathcal{S}} Rg \prod_{M=1,2} dx^M dy^M \tag{5.2}$$

where x^M and y^M are the real and imaginary parts respectively of the joint coordinates z^M and g denotes the determinant of the Dirac metric $g_{\dot{M}N}$. The measure is invariant under chiral coordinate transformations, which leave (5.1) invariant, as may be seen as follows. Under a general chiral coordinate transformation the determinant of the metric transforms according to $g' = g\det(K_r\bar{K}_l)^{-1}$, where the holomorphic coordinate transformation matrices are given by $K_i^{M'}{}_N = \partial z_i^{M'}/\partial z_i^N$ and $z_i^{M'}$ denotes the new coordinates. In the special case of a transformation, which leaves the condition (5.1) invariant, K_r and K_l are equal and $\det(K_r\bar{K}_l)$ becomes real; as a consequence of the Cauchy-Riemann conditions this determinant equals the Jacobian determinant of $\prod_M dx^M dz^M$.

The Lagrangean density

$$\Lambda = gR_{\dot{M}N}g^{N\dot{M}} + \bar{g}R^+_{\dot{M}N}g^{+N\dot{M}} \tag{5.3}$$

is varied with respect to the Dirac metric. Since the second term depends only on the adjoint Dirac metric, it does not contribute to the variation of Λ with respect to the Dirac metric:

$$\delta\Lambda = g[g^{N\dot{M}}\delta R_{\dot{M}N} + R_{\dot{M}N}\delta g^{N\dot{M}}] - gRg_{\dot{M}N}\delta g^{N\dot{M}} \tag{5.4}$$

The last term in this expression arises from the variation of the determinant g. The variation of the Ricci spinor may be derived from the explicit expression of the curvature spinor in terms of the Dirac metric

$$\begin{aligned}\delta R_{\dot{M}N} = \delta R_r^S{}_{N\dot{M}S} &= \partial^l_{\dot{M}}\delta(g^{S\dot{R}}\partial^r_S g_{\dot{R}N}) = \partial^l_{\dot{M}}[g^{S\dot{R}}\partial^r_S\delta g_{\dot{R}N} - g^{S\dot{Q}}\delta g_{\dot{Q}P}(g^{P\dot{R}}\partial^r_S g_{\dot{R}N})]\\ &= \partial^l_{\dot{M}}[g^{S\dot{R}}\mathrm{D}^r_S\delta g_{\dot{R}N}] = \mathrm{D}^l_{\dot{M}}[g^{S\dot{R}}\mathrm{D}^r_S\delta g_{\dot{R}N}]\end{aligned} \tag{5.5}$$

where D^i_M denotes covariant derivation in the direction of the coordinate z_i^M. Using further the compatibility of the connection with the Dirac metric, the first term in the variation of the Lagrangean density reads:

$$gg^{N\dot{M}}\delta R_{\dot{M}N} = -g\mathrm{D}^l_{\dot{M}}\mathrm{D}^r_N\delta g^{N\dot{M}} \tag{5.6}$$

The covariant divergences may be replaced with coordinate divergences and torsion terms

$$gg^{N\dot{M}}\delta R_{\dot{M}N} = g[-i\mathrm{D}^r_N\varphi_{\dot{M}} + \chi^+_N\varphi_{\dot{M}}]\delta g^{N\dot{M}} - \partial^l_{\dot{M}}(g\mathrm{D}^r_N\delta g^{N\dot{M}}) + i\,\partial^r_N(g\varphi_{\dot{M}}\delta g^{N\dot{M}}) \tag{5.7}$$

where the following identities have been used, which are valid for arbitrary spinor fields $X^M \in W(M_r)$ and $Y^{\dot{M}} \in \bar{W}(M_l)$:

$$gD^r_M X^M = \partial^r_M (gX^M) - ig\chi^+_M X^M, \qquad gD^l_{\dot{M}} Y^{\dot{M}} = \partial^l_{\dot{M}}(gY^{\dot{M}}) - ig\varphi_{\dot{M}} Y^{\dot{M}} \tag{5.8}$$

The variation of the Lagrangean density finally takes the form:

$$\delta\Lambda = g[R_{\dot{M}N} - g_{\dot{M}N} R - iD^r_N \varphi_{\dot{M}} + \chi^+_N \varphi_{\dot{M}}]\delta g^{N\dot{M}} - \partial^l_{\dot{M}}(gD^r_N \delta g^{N\dot{M}}) + i\partial^r_N (g\varphi_{\dot{M}} \delta g^{N\dot{M}}) \tag{5.9}$$

The last two terms on the right hand side of (5.9) yield only boundary terms in the variation of the action integral and will not be considered further. The requirement of stationary action forces the expression in brackets to vanish, yielding the *spinor Einstein equation:*

$$R_{\dot{A}B} - Rg_{\dot{A}B} = iD^r_B \varphi_{\dot{A}} - \chi^+_B \varphi_{\dot{A}} \tag{5.10}$$

From the trace of the spinor Einstein equation an expression for the curvature scalar R in terms of the spinor field is obtained:

$$R = \chi^+ \varphi - iD^r_A \varphi^A \tag{5.11}$$

One may also vary the Lagrangean density with respect to the adjoint Dirac metric, which yields the following equation:

$$\widetilde{R}_{\dot{A}B} - Rg_{\dot{A}B} = iD^l_{\dot{A}} \chi^+_B - \chi^+_B \varphi_{\dot{A}} \tag{5.12}$$

Since the action is real, this equation must not be independent of the spinor Einstein equation. It will indeed be seen in the following that the two equations are equivalent as a consequence of the second Bianchi identities.

5.2 Field Tensors

The dynamics of spinor relativity may also be expressed in form of tensor field equations. To accomplish this, the Ricci spinors are related to the Dirac-Ricci tensor and the complex electromagnetic field as follows. In (3.27) the spinor basis may be replaced by a Minkowski basis:

$$\Omega^A_{r\ B} = \frac{1}{4} R^A_{r\ B\dot{C}D} \zeta^{\dot{C}}_\alpha \xi^D_\beta\, \vartheta^\alpha \wedge \vartheta^\beta \tag{5.13}$$

On the other hand, an expansion of the right curvature matrix in a Minkowski basis is also obtained from (4.52) and (4.55):

$$\Omega^A_{r\ B} = \frac{1}{2}[-i\mathcal{F}_{\alpha\beta}\delta^A{}_B + Q^k_{\alpha\beta}\sigma_k{}^A{}_B]\, \vartheta^\alpha \wedge \vartheta^\beta \tag{5.14}$$

Comparison of these expressions yields:

$$\mathcal{F}_{\alpha\beta} = \frac{i}{4} R^C_{r\ C\dot{A}B}\, \zeta^{\dot{A}}_{[\alpha} \xi^B_{\beta]}, \qquad Q^k_{\alpha\beta} = \frac{1}{4}\sigma_k{}^C{}_D R^D_{r\ C\dot{A}B}\, \zeta^{\dot{A}}_{[\alpha} \xi^B_{\beta]} \tag{5.15}$$

In order to obtain the Dirac-Ricci tensor, (5.15) is inserted into (4.57) and the resulting expressions are simplified with help of (4.24) and (4.30):

$$I_\alpha{}^\gamma \mathcal{F}_{\gamma\beta} = -\frac{1}{4} R^C_{r\ C\dot{A}B} \zeta^{\dot{A}}_{(\alpha} \xi^B_{\beta)}, \qquad \Sigma^{k\gamma}_\alpha Q^k_{\gamma\beta} = \frac{1}{8}\sigma_k{}^C{}_D R^D_{r\ C\dot{A}B}\, [\zeta^{\dot{E}}_\alpha \sigma^k{}_{\dot{E}}{}^{\dot{A}} \xi^B_\beta + \zeta^{\dot{A}}_\beta \sigma_k{}^B{}_E \xi^E_\alpha] \tag{5.16}$$

The products of Pauli matrices may further be evaluated using the Fiertz identities (4.17):

$$4\Sigma_\alpha^{k\gamma} Q^k_{\gamma\beta} = R^r_{\dot{E}C}{}^C{}_B\, \zeta^{\dot{E}}_\alpha \xi^B_\beta + R^B_r{}_{C\dot{A}B}\, \zeta^{\dot{A}}_\beta \xi^C_\alpha - R^C_r{}_{C\dot{A}B}\, \zeta^{\dot{A}}_{(\alpha}\xi^B_{\beta)} \tag{5.17}$$

Comparison with the definition (3.29) of the Ricci spinors finally yields the desired expressions for the Dirac-Ricci tensor and the complex electromagnetic field in terms of Ricci spinors:

$$\mathcal{F}_{\alpha\beta} = \frac{i}{4}\rho_{\dot{A}B}\, \zeta^{\dot{A}}_{[\alpha}\xi^B_{\beta]}, \qquad \mathcal{R}_{\alpha\beta} = -\frac{1}{4}[R_{\dot{C}D}\, \zeta^{\dot{C}}_\beta \xi^D_\alpha + \widetilde{R}_{\dot{C}D}\, \zeta^{\dot{C}}_\alpha \xi^D_\beta] \tag{5.18}$$

From these expressions it is seen with help of (4.30) that both tensors commute with the complex spinor structure I.

For connections with torsion one may define the *axial Ricci tensor*, which is the contracted dual curvature tensor:

$$\widetilde{\mathcal{R}}_{\alpha\beta} = \widetilde{\mathcal{R}}^\gamma{}_{\alpha\gamma\beta}, \qquad \widetilde{\mathcal{R}}_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\kappa\lambda}\mathcal{R}_{\kappa\lambda\gamma\delta} \tag{5.19}$$

The axial Ricci tensor of the Dirac connection is however not independent, but may be expressed in terms of the Dirac-Ricci tensor and the complex electromagnetic field, as follows from (4.56) and the (anti-)selfduality of I and Σ_k:

$$i\widetilde{\mathcal{R}}_{\alpha\beta} = \mathcal{R}_{\alpha\beta} - 2(I\mathcal{F})_{\alpha\beta} \tag{5.20}$$

Using the identity (4.32) it is further seen with help of (4.30) that the dual of the complex electromagnetic field may be written as follows:

$$i\mathcal{F}^*_{\alpha\beta} = \mathcal{F}_{\alpha\beta} - \frac{1}{4}\rho I_{\alpha\beta} \tag{5.21}$$

From the Dirac-Ricci tensor and the axial Ricci tensor one obtains the *Dirac-Einstein tensor* and the *axial Einstein tensor* by subtraction of a trace term or equivalently by contraction of the 'double dual' of the corresponding curvature tensor:

$$\mathcal{G}_{\alpha\beta} = \mathcal{R}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\mathcal{R}^\gamma{}_\gamma = \widetilde{\mathcal{R}}^{*\gamma}{}_{\beta\gamma\alpha}, \qquad \widetilde{\mathcal{G}}_{\alpha\beta} = \widetilde{\mathcal{R}}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\widetilde{\mathcal{R}}^\gamma{}_\gamma = -\mathcal{R}^{*\gamma}{}_{\beta\gamma\alpha} \tag{5.22}$$

where the asterisk denotes the dual with respect to the second index pair and the minus sign in the last term arises from taking twice the dual with respect to the first index pair. The traces of the field tensors are derived with help of (4.28)

$$\mathcal{G}^\gamma{}_\gamma = -\mathcal{R}^\gamma{}_\gamma = R, \qquad \widetilde{\mathcal{G}}^\gamma{}_\gamma = -\widetilde{\mathcal{R}}^\gamma{}_\gamma = i(\rho - R) \tag{5.23}$$

while contraction with the complex spinor structure yields:

$$(\mathcal{F}I)^\gamma{}_\gamma = -i(\mathcal{F}^*I)^\gamma{}_\gamma = -\frac{1}{2}\rho, \qquad (\mathcal{R}I)^\gamma{}_\gamma = (\widetilde{\mathcal{R}}I)^\gamma{}_\gamma = 0 \tag{5.24}$$

5.3 Second Bianchi Identities

From the second Bianchi identities (3.13) the following relations involving the Riemann spinors and covariant derivatives of the torsion spinors are derived

$$\mathrm{D}^i_{[C} T_i^D{}_{AB]} + T_i^D{}_{E[C} T_i^E{}_{AB]} = 0, \qquad \mathrm{D}^{\tilde{i}}_{\dot{A}} T_i^D{}_{CB} = R_i^D{}_{B\dot{A}C} - R_i^D{}_{C\dot{A}B} \tag{5.25}$$

where $\tilde{i}$ is the chirality index opposite to i. These two equations are the $(3_i, 0)$-type and $(2_i, 1_{\tilde{i}})$-type part respectively of the 3-form identities (3.13). In case of a Dirac manifold of dimension $n = 2$ the left hand side of the first equation vanishes identically since the are no antisymmetric combinations of three indices. Contraction of the second equation yields the *magnetic identities*

$$-i\mathrm{D}^l_{\dot{A}}\chi^+_B = R_{\dot{A}B} - \rho_{\dot{A}B}, \qquad -i\mathrm{D}^r_B\varphi_{\dot{A}} = \widetilde{R}_{\dot{A}B} - \rho_{\dot{A}B} \tag{5.26}$$

where the hermitian adjoint was taken in the case $i = l$. With help of the magnetic identities it is seen that the dynamical equation (5.12), obtained from varying the action with respect to the adjoint Dirac metric, is equivalent to the spinor Einstein equation. Further, the right spinor divergence is seen to be complex conjugate to the left spinor divergence and related to the curvature scalars as follows:

$$\mathrm{D}^r_A\varphi^A = \mathrm{D}^l_{\dot{A}}\chi^{\dot{A}} = i(R-\rho) \tag{5.27}$$

The second Bianchi identity may also be written in tensor form:

$$d\tau^\alpha + \Theta^\alpha{}_\beta \wedge \tau^\beta = \Omega^\alpha{}_\beta \wedge \vartheta^\beta \tag{5.28}$$

The $(3,0)_{ch}$-part and the $(2,1)_{ch}$-part of this 3-form identity yield the following tensor equations

$$\mathrm{D}^{ch}_{[\delta}T^\alpha{}_{\beta\gamma]} + T^\alpha{}_{\epsilon[\delta}T^\epsilon{}_{\beta\gamma]} = \mathcal{R}^\alpha{}_{[\beta\gamma\delta]}, \qquad \bar{\mathrm{D}}^{ch}_\delta T^\alpha{}_{\beta\gamma} = 0 \tag{5.29}$$

where D^{ch}_α denotes covariant derivation in the direction of $\mathcal{E}_\alpha$. The first equation may be contracted with the totally antisymmetric tensor, which is covariantly constant as follows from its representation (4.32). This yields an identity for the axial Einstein tensor

$$\widetilde{\mathcal{G}}_{\alpha\beta} = -\mathrm{D}^{ch}_\gamma T^{*}_{\beta\alpha}{}^\gamma + T^{*}_{\gamma\delta\alpha}T_\beta{}^{\gamma\delta}, \qquad \bar{\mathrm{D}}^{ch}_\delta T^\alpha{}_{\beta\gamma} = 0 \tag{5.30}$$

where the asterisk denotes the dual torsion tensor; the torsion tensor of the Dirac connection is selfdual, as may be derived from (4.63):

$$T^*_{\alpha\beta\gamma} = \frac{1}{2}T_{\alpha\delta\epsilon}\epsilon^{\delta\epsilon}{}_{\beta\gamma} = iT_{\alpha\beta\gamma} \tag{5.31}$$

One further obtains the following properties of the torsion of the Dirac connection

$$T^\beta{}_{\alpha\beta} = iT_\alpha, \qquad T^\beta{}_{\alpha\gamma}I^\gamma{}_\beta = J_\alpha, \qquad T^\alpha{}_{\beta\gamma}I^{\beta\gamma} = 0, \qquad T_{\gamma\delta\alpha}T_\beta{}^{\gamma\delta} = 0 \tag{5.32}$$

which are used to derive the *contracted second Bianchi identities* from (5.30)

$$\mathrm{D}^{ch}_\alpha T^\alpha = i(R-\rho), \qquad \mathrm{D}^{ch}_\alpha J^\alpha = 0; \qquad \bar{\mathrm{D}}^{ch}_\alpha T_\beta = 0, \qquad \bar{\mathrm{D}}^{ch}_\alpha J_\beta = 0 \tag{5.33}$$

where the relations (5.23) and (5.24) have been taken into account.

From these identities together with the contracted spinor Einstein equation (5.11) it is seen that the curvature scalars are related to the scalars of the fundamental spinor field as follows:

$$R = \mu^2 + \sigma, \qquad \rho = \mu^2; \qquad \mu^2 = \chi^+\varphi = T_\alpha T^\alpha, \qquad i\sigma = \mathrm{D}^r_A\varphi^A = \mathrm{D}^{ch}_\alpha T^\alpha \tag{5.34}$$

5.4 Einstein Equations

In order to derive tensor Einstein equations, the spinor Einstein equation (5.10) and the magnetic identities (5.26) are inserted into (5.18):

$$\begin{aligned}\mathcal{R}_{\alpha\beta} &= -\frac{i}{4}[\,\mathrm{D}^l_{\dot{C}}\chi^+_D\,\zeta^{\dot{C}}_\alpha\xi^D_\beta + \mathrm{D}^r_D\varphi_{\dot{C}}\,\zeta^{\dot{C}}_\beta\xi^D_\alpha] + \frac{1}{2}[\chi^+_B\varphi_{\dot{A}} - \delta_{\dot{A}B}(\mu^2+\sigma)]\zeta^{\dot{A}}_{(\alpha}\xi^B_{\beta)}\\ \mathcal{F}_{\alpha\beta} &= -\frac{1}{4}[\mathrm{D}^l_{\dot{A}}\chi^+_B + \mathrm{D}^r_B\varphi_{\dot{A}}]\zeta^{\dot{A}}_{[\alpha}\xi^B_{\beta]} - \frac{i}{4}[\chi^+_B\varphi_{\dot{A}} - \delta_{\dot{A}B}(\mu^2+\sigma)]\zeta^{\dot{A}}_{[\alpha}\xi^B_{\beta]}\end{aligned} \tag{5.35}$$

The Dirac metric takes unit form, since the indices refer to a spinor basis. The covariant spinor derivatives may be expressed in terms of vector derivatives with help of (4.30):

$$\xi^C_\alpha\mathrm{D}^r_C = (\eta^\beta{}_\alpha - iI^\beta{}_\alpha)\mathrm{D}^{ch}_\beta, \qquad \zeta^{\dot{C}}_\alpha\mathrm{D}^l_{\dot{C}} = (\eta^\beta{}_\alpha + iI^\beta{}_\alpha)\mathrm{D}^{ch}_\beta \tag{5.36}$$

Using further the relations (4.29) in the last terms, the Einstein equations may be written in the form:

$$\begin{aligned}\mathcal{R}_{\alpha\beta} &= -\frac{i}{4}[(\eta+iI)^\gamma{}_\alpha\xi^B_\beta\mathrm{D}^{ch}_\gamma\chi^+_B + (\eta-iI)^\gamma{}_\alpha\zeta^{\dot{B}}_\beta\mathrm{D}^{ch}_\gamma\varphi_{\dot{B}}] + \frac{1}{2}\chi^+_B\varphi_{\dot{A}}\zeta^{\dot{A}}_{(\alpha}\xi^B_{\beta)}\\ &\quad - \frac{1}{2}(\mu^2+\sigma)\eta_{\alpha\beta}\\ \mathcal{F}_{\alpha\beta} &= -\frac{1}{4}[(\eta+iI)^\gamma{}_{[\alpha}\xi^B_{\beta]}\mathrm{D}^{ch}_\gamma\chi^+_B + (\eta-iI)^\gamma{}_{[\beta}\zeta^{\dot{A}}_{\alpha]}\mathrm{D}^{ch}_\gamma\varphi_{\dot{A}}] - \frac{i}{4}\chi^+_B\varphi_{\dot{A}}\zeta^{\dot{A}}_{[\alpha}\xi^B_{\beta]}\\ &\quad + \frac{1}{4}(\mu^2+\sigma)I_{\alpha\beta}\end{aligned} \tag{5.37}$$

The expressions on the right hand sides may be simplified using the vector fields (4.61) and taking into account the covariant constancy of the spinor tetrad fields:

$$\begin{aligned}\mathcal{R}_{\alpha\beta} &= -\frac{i}{2}(\mathrm{D}^{ch}_\gamma T_\delta)(\eta^\gamma{}_\alpha\eta^\delta{}_\beta + I^\gamma{}_\alpha I^\delta{}_\beta) + \frac{1}{2}(T_\alpha T_\beta - J_\alpha J_\beta) - \frac{1}{2}(\mu^2+\sigma)\eta_{\alpha\beta}\\ \mathcal{F}_{\alpha\beta} &= \frac{1}{2}(\mathrm{D}^{ch}_\gamma J_\delta)(\eta^\gamma{}_{[\alpha}\eta^\delta{}_{\beta]} + I^\gamma{}_{[\alpha}I^\delta{}_{\beta]}) - \frac{i}{2}J_{[\alpha}T_{\beta]} + \frac{1}{4}(\mu^2+\sigma)I_{\alpha\beta}\end{aligned} \tag{5.38}$$

In the first of these equations the Dirac-Ricci tensor is replaced with the Dirac-Einstein tensor (5.22) yielding the *gravitational Einstein equation:*

$$\mathcal{G}_{\alpha\beta} = -\frac{i}{2}(\mathrm{D}^{ch}_\gamma T_\delta)(\eta^\gamma{}_\alpha\eta^\delta{}_\beta + I^\gamma{}_\alpha I^\delta{}_\beta) + \frac{1}{2}(T_\alpha T_\beta - J_\alpha J_\beta) \tag{5.39}$$

Using the identity (4.32), the *electromagnetic Einstein equation* may be written in the form:

$$\mathcal{F}_{\alpha\beta} = \frac{1}{2}(\mathrm{D}^{ch}_\gamma J_\delta)\left(\eta^\gamma{}_{[\alpha}\eta^\delta{}_{\beta]} + \frac{i}{2}\epsilon^{\gamma\delta}{}_{\alpha\beta}\right) - \frac{i}{2}J_{[\alpha}T_{\beta]} + \frac{1}{4}\mu^2 I_{\alpha\beta} \tag{5.40}$$

Inserting these equations into (5.20) one further obtains an *axial Einstein equation*, which however does not provide independent information:

$$i\widetilde{\mathcal{G}}_{\alpha\beta} = \frac{i}{2}(\mathrm{D}^{ch}_\gamma T_\delta)(\eta^\gamma{}_\beta\eta^\delta{}_\alpha + I^\gamma{}_\beta I^\delta{}_\alpha) + \frac{1}{2}\sigma\eta_{\alpha\beta} \tag{5.41}$$

5.5 First Bianchi Identities

In addition to the dynamical Einstein equations, further information is obtained from the first Bianchi identities. Taking the trace of (3.11) it is seen that the electromagnetic 2-form is closed, which also follows immediately from (4.54):

$$d\mathcal{F} = 0 \tag{5.42}$$

In order to reexpress this in terms of the covariant divergence of the dual electromagnetic field tensor, it is more convenient to start from the first Bianchi identity written in tensor form:

$$d\Omega^{\alpha}{}_{\beta} + \Theta^{\alpha}{}_{\gamma} \wedge \Omega^{\gamma}{}_{\beta} - \Omega^{\alpha}{}_{\gamma} \wedge \Theta^{\gamma}{}_{\beta} = 0 \tag{5.43}$$

Expanding the matrix valued forms in a Minkowski basis one obtains:

$$\mathrm{D}^{ch}_{[\epsilon}\mathcal{R}^{\alpha\beta}{}_{\gamma\delta]} + \mathcal{R}^{\alpha\beta}{}_{\kappa[\epsilon}T^{\kappa}{}_{\gamma\delta]} = 0, \qquad \bar{\mathrm{D}}^{ch}_{\epsilon}\mathcal{R}^{\alpha}{}_{\beta\gamma\delta} = 0 \tag{5.44}$$

These equations are the $(3,0)_{ch}$-type part and $(2,1)_{ch}$-type part respectively of the 3-form identity (5.43). The first of these relations may be written in a more convenient form by contracting it with the totally antisymmetric tensor

$$\mathrm{D}^{ch}_{\delta}\mathcal{R}^{*}_{\alpha\beta\gamma}{}^{\delta} + \frac{1}{2}\mathcal{R}^{*}_{\alpha\beta\delta\epsilon}\widehat{T}_{\gamma}{}^{\delta\epsilon} = 0, \qquad \bar{\mathrm{D}}^{ch}_{\epsilon}\mathcal{R}^{*}_{\alpha\beta\gamma\delta} = 0 \tag{5.45}$$

where $\widehat{T}$ is a modified torsion tensor with trace terms subtracted; in case of the Dirac connection the modified torsion tensor differs from (4.63) by the sign between its two contributions:

$$\widehat{T}^{\alpha}{}_{\beta\gamma} = T^{\alpha}{}_{\beta\gamma} + \eta^{\alpha}{}_{\beta}T^{\delta}{}_{\gamma\delta} - \eta^{\alpha}{}_{\gamma}T^{\delta}{}_{\beta\delta} = I^{\alpha}{}_{[\beta}J_{\gamma]} + i\eta^{\alpha}{}_{[\beta}T_{\gamma]} \tag{5.46}$$

Contracting (5.45) with I and taking into account its covariant constancy, the following identities for the dual electromagnetic field tensor are obtained, which are equivalent to (5.42):

$$\mathrm{D}^{ch}_{\beta}\mathcal{F}^{*\beta}_{\alpha} + \frac{1}{2}\mathcal{F}^{*}_{\beta\gamma}\widehat{T}_{\alpha}{}^{\beta\gamma} = 0, \qquad \bar{\mathrm{D}}^{ch}_{\alpha}\mathcal{F}^{*}_{\beta\gamma} = 0 \tag{5.47}$$

Taking the dual with respect to its first index pair of the curvature tensor in (5.45) before contracting with the Minkowski metric yields the corresponding identities for the Dirac-Einstein tensor:

$$\mathrm{D}^{ch}_{\beta}\mathcal{G}^{\beta}{}_{\alpha} - \frac{1}{2}\widetilde{\mathcal{R}}^{*}_{\alpha\beta\gamma\delta}\widehat{T}^{\beta\gamma\delta} = 0, \qquad \bar{\mathrm{D}}^{ch}_{\alpha}\mathcal{G}_{\beta\gamma} = 0 \tag{5.48}$$

Inserting the expression (5.46) for the modified torsion tensor of the Dirac connection into (5.47) and (5.48) and taking into account that the Dirac-Einstein tensor as well as the dual electromagnetic field tensor commute with I, the *contracted first Bianchi identities* simplify as follows:

$$(\mathrm{D}^{ch}_{\beta} + iT_{\beta})\mathcal{F}^{*\beta}_{\alpha} = 0, \qquad \bar{\mathrm{D}}^{ch}_{\alpha}\mathcal{F}^{*}_{\beta\gamma} = 0; \qquad (\mathrm{D}^{ch}_{\beta} + iT_{\beta})\mathcal{G}^{\beta}{}_{\alpha} = 0, \qquad \bar{\mathrm{D}}^{ch}_{\alpha}\mathcal{G}_{\beta\gamma} = 0 \tag{5.49}$$

6 Heisenberg Equation System

In this chapter the field equations of spinor relativity are reformulated in such a way that they resemble more closely the usual Einstein, Maxwell and Dirac equations on spacetime describing gravitational and electromagnetic fields coupled to a Dirac spinor field. To this end the complex Minkowski bases are decomposed into their real and imaginary parts, defining classical and axial vectors respectively. As a consequence of the chiral Kähler condition derivatives in the directions of axial vectors may be expressed in terms of derivatives in classical vector directions. In order to describe the gravitational field, the Einstein connection ∇ is introduced, which is obtained from the Dirac connection D by removing its torsion. The Heisenberg equation system consists of gravitational and electromagnetic Einstein equations expressed in terms of the Einstein connection, both homogeneous and inhomogeneous Maxwell equations for the complex electromagnetic field and a non-linear generalized Dirac equation for the fundamental spinor field.

6.1 Real Minkowski Bases

In order to define real Minkowski bases in the tangent bundle, the complex basis vectors are decomposed into their real and imaginary parts:

$$e_\alpha = \mathcal{E}_\alpha + \bar{\mathcal{E}}_\alpha, \qquad \tilde{e}_\alpha = -i(\mathcal{E}_\alpha - \bar{\mathcal{E}}_\alpha) \tag{6.1}$$

The real basis forms dual to e_α and $\tilde{e}_\alpha$ are given by:

$$\omega^\alpha = \frac{1}{2}(\vartheta^\alpha + \bar{\vartheta}^\alpha), \qquad \tilde{\omega}^\alpha = \frac{i}{2}(\vartheta^\alpha - \bar{\vartheta}^\alpha) \tag{6.2}$$

The linear combinations of the basis fields e_α are called *classical vectors*, while the linear combinations of the basis fields $\tilde{e}_\alpha$ define *axial vectors*. The spaces of classical and axial vectors are Lorentz invariant but $\mathbf{D}(1)$-gauge transformations mix them. Between real basis vectors the product in the tangent bundle takes the form:

$$(e_\alpha, e_\beta) = -(\tilde{e}_\alpha, \tilde{e}_\beta) = \eta_{\alpha\beta}, \qquad (e_\alpha, \tilde{e}_\beta) = (\tilde{e}_\alpha, e_\beta) = 0 \tag{6.3}$$

The structure equations for real bases are derived from the corresponding equation (4.40) for complex Minkowski bases

$$\begin{aligned} d\omega^\alpha &= -\frac{1}{2}(\mathrm{Re}\, c^\alpha{}_{\beta\gamma}\omega^\beta \wedge \omega^\gamma + \mathrm{Im}\, c^\alpha{}_{\beta\gamma}\tilde{\omega}^\beta \wedge \omega^\gamma + \mathrm{Re}\, \tilde{c}^\alpha{}_{\beta\gamma}\omega^\beta \wedge \tilde{\omega}^\gamma \\ &\quad + \mathrm{Im}\, \tilde{c}^\alpha{}_{\beta\gamma}\tilde{\omega}^\beta \wedge \tilde{\omega}^\gamma) \\ d\tilde{\omega}^\alpha &= -\frac{1}{2}(\mathrm{Re}\, \tilde{c}^\alpha{}_{\beta\gamma}\tilde{\omega}^\beta \wedge \tilde{\omega}^\gamma - \mathrm{Im}\, \tilde{c}^\alpha{}_{\beta\gamma}\omega^\beta \wedge \tilde{\omega}^\gamma + \mathrm{Re}\, c^\alpha{}_{\beta\gamma}\tilde{\omega}^\beta \wedge \omega^\gamma \\ &\quad - \mathrm{Im}\, c^\alpha{}_{\beta\gamma}\omega^\beta \wedge \omega^\gamma) \end{aligned} \tag{6.4}$$

with new structure functions given by linear combinations of $\mathcal{C}$ and $\widetilde{\mathcal{C}}$

$$\begin{aligned} c^\alpha{}_{\beta\gamma} &= (\mathcal{C} + \widetilde{\mathcal{C}})^\alpha{}_{\beta\gamma} = \xi^M_\beta \partial_\gamma \xi^\alpha_M + \zeta^{\dot{M}}_\beta \partial_\gamma \zeta^\alpha_{\dot{M}}, \\ \tilde{c}^\alpha{}_{\beta\gamma} &= -i(\mathcal{C} - \widetilde{\mathcal{C}})^\alpha{}_{\beta\gamma} = \xi^M_\beta \tilde{\partial}_\gamma \xi^\alpha_M + \zeta^{\dot{M}}_\beta \tilde{\partial}_\gamma \zeta^\alpha_{\dot{M}} \end{aligned} \tag{6.5}$$

where ∂_α and $\tilde{\partial}_\alpha$ denote derivatives in the directions of the basis vectors e_α and $\tilde{e}_\alpha$ respectively. Equivalently one obtains the commutators of basis vectors, which show that in general the subspaces of classical or axial vectors are not integrable:

$$\begin{aligned} &[e_\alpha, e_\beta] = \operatorname{Re} c^\gamma{}_{[\alpha\beta]} e_\gamma - \operatorname{Im} c^\gamma{}_{[\alpha\beta]} \tilde{e}_\gamma, \qquad [\tilde{e}_\alpha, \tilde{e}_\beta] = \operatorname{Re} \tilde{c}^\gamma{}_{[\alpha\beta]} \tilde{e}_\gamma + \operatorname{Im} \tilde{c}^\gamma{}_{[\alpha\beta]} e_\gamma \\ &[\tilde{e}_\alpha, e_\beta] = \frac{1}{2}(\operatorname{Im} c^\gamma{}_{\alpha\beta} - \operatorname{Re} \tilde{c}^\gamma{}_{\beta\alpha}) e_\gamma + \frac{1}{2}(\operatorname{Re} c^\gamma{}_{\alpha\beta} + \operatorname{Im} \tilde{c}^\gamma{}_{\beta\alpha}) \tilde{e}_\gamma \end{aligned} \tag{6.6}$$

The Dirac connection mixes classical and axial vectors, as may be seen from the connection matrix in terms of real bases:

$$\begin{aligned} &\mathrm{D}(e_\alpha, \tilde{e}_\alpha) = (e_\beta, \tilde{e}_\beta) \begin{pmatrix} \operatorname{Re} \Theta^\beta{}_\alpha & \operatorname{Im} \Theta^\beta{}_\alpha \\ -\operatorname{Im} \Theta^\beta{}_\alpha & \operatorname{Re} \Theta^\beta{}_\alpha \end{pmatrix}, \\ &\mathrm{D} \begin{pmatrix} \omega^\alpha \\ \tilde{\omega}^\alpha \end{pmatrix} = - \begin{pmatrix} \operatorname{Re} \Theta^\alpha{}_\beta & \operatorname{Im} \Theta^\alpha{}_\beta \\ -\operatorname{Im} \Theta^\alpha{}_\beta & \operatorname{Re} \Theta^\alpha{}_\beta \end{pmatrix} \begin{pmatrix} \omega^\beta \\ \tilde{\omega}^\beta \end{pmatrix} \end{aligned} \tag{6.7}$$

6.2 Einstein Connection for Vectors

The gravitational field is described by the torsion-free *Einstein connection* ∇. Its connection matrix in a Minkowski basis is obtained from the Dirac connection matrix (4.50) by subtracting the contortion tensor from the connection coefficients (4.65):

$$(\Theta_\mathcal{E})^\alpha{}_\beta = \Theta^\alpha{}_\beta - \kappa^\alpha{}_\beta, \qquad \kappa^\alpha{}_\beta = \mathcal{K}^\alpha{}_{\beta\gamma} \vartheta^\gamma \tag{6.8}$$

The corresponding curvature matrix is given by:

$$\begin{aligned} &(\Omega_\mathcal{E})^\alpha{}_\beta = \Omega^\alpha{}_\beta - \Xi^\alpha{}_\beta, \qquad (\Omega_\mathcal{E})^\alpha{}_\beta = d(\Theta_\mathcal{E})^\alpha{}_\beta + (\Theta_\mathcal{E})^\alpha{}_\gamma \wedge (\Theta_\mathcal{E})^\gamma{}_\beta \\ &\Xi^\alpha{}_\beta = d\kappa^\alpha{}_\beta + \Theta^\alpha{}_\gamma \wedge \kappa^\gamma{}_\beta + \kappa^\alpha{}_\gamma \wedge \Theta^\gamma{}_\beta - \kappa^\alpha{}_\gamma \wedge \kappa^\gamma{}_\beta \end{aligned} \tag{6.9}$$

An expansion of the Einstein connection matrix in terms of real basis forms yields

$$(\Theta_\mathcal{E})^\alpha{}_\beta = \Gamma^\alpha{}_{\beta\gamma} \omega^\gamma + \widetilde{\Gamma}^\alpha{}_{\beta\gamma} \tilde{\omega}^\gamma \tag{6.10}$$

where the connection coefficients are obtained from comparison with (4.50):

$$\Gamma^\alpha{}_{\beta\gamma} = \Delta^\alpha{}_{\beta\gamma} - \mathcal{K}^\alpha{}_{\beta\gamma} - \frac{1}{2}\widetilde{\mathcal{C}}^\alpha{}_{\beta\gamma}, \qquad \widetilde{\Gamma}^\alpha{}_{\beta\gamma} = -i\left(\Delta^\alpha{}_{\beta\gamma} - \mathcal{K}^\alpha{}_{\beta\gamma} + \frac{1}{2}\widetilde{\mathcal{C}}^\alpha{}_{\beta\gamma}\right) \tag{6.11}$$

Since they are antisymmetric in their first index pair, they may be written in a form analogous to (4.65), involving the structure functions c and $\tilde{c}$:

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(c_{\gamma[\alpha\beta]} + c_{\beta[\alpha\gamma]} - c_{\alpha[\beta\gamma]}), \qquad \widetilde{\Gamma}_{\alpha\beta\gamma} = \frac{1}{2}(\tilde{c}_{\gamma[\alpha\beta]} + \tilde{c}_{\beta[\alpha\gamma]} - \tilde{c}_{\alpha[\beta\gamma]}) \tag{6.12}$$

The expansion of the matrix of 2-forms Ξ in terms of Minkowski basis forms is given by:

$$\Xi^\alpha{}_\beta = \left[\mathrm{D}^{ch}_\gamma \mathcal{K}^\alpha{}_{\beta\delta} + \frac{1}{2}\mathcal{K}^\alpha{}_{\beta\epsilon} T^\epsilon{}_{\gamma\delta} - \mathcal{K}^\alpha{}_{\epsilon\gamma}\mathcal{K}^\epsilon{}_{\beta\delta}\right] \vartheta^\gamma \wedge \vartheta^\delta + [\bar{\mathrm{D}}^{ch}_\gamma \mathcal{K}^\alpha{}_{\beta\delta}]\, \bar{\vartheta}^\gamma \wedge \vartheta^\delta \tag{6.13}$$

The last term vanishes as a consequence of the second Bianchi identity (5.29). Thus Ξ is a matrix of forms of type $(2,0)_{ch}$ and consequently the same is true for the Einstein curvature matrix $\Omega_{\mathcal{E}}$. The expansion of $\Omega_{\mathcal{E}}$ defines the *Riemann tensor*

$$(\Omega_{\mathcal{E}})^{\alpha}{}_{\beta} = \frac{1}{2} R^{\alpha}{}_{\beta\gamma\delta}\, \vartheta^{\gamma} \wedge \vartheta^{\delta} \tag{6.14}$$

which in terms of connection coefficients is given by:

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_{\gamma}\Gamma^{\alpha}{}_{\beta\delta} - \partial_{\delta}\Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\beta\epsilon}\mathrm{Re}\, c^{\epsilon}{}_{[\gamma\delta]} + \Gamma^{\alpha}{}_{\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\epsilon\delta}\Gamma^{\epsilon}{}_{\beta\gamma} \tag{6.15}$$

Contraction of the Riemann tensor yields the *Ricci tensor*. Its relation to the Dirac-Ricci tensor is obtained from (6.13) as follows:

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta} = \mathcal{R}_{\alpha\beta} - [\mathrm{D}^{ch}_{\gamma}\mathcal{K}^{\gamma}{}_{\alpha\beta} - \mathrm{D}^{ch}_{\beta}\mathcal{K}^{\gamma}{}_{\alpha\gamma} + \mathcal{K}^{\gamma}{}_{\alpha\delta}\mathcal{K}^{\delta}{}_{\beta\gamma} - \mathcal{K}^{\delta}{}_{\gamma\delta}\mathcal{K}^{\gamma}{}_{\alpha\beta}] \tag{6.16}$$

From the expression (4.66) of the contortion tensor in terms of the vector fields the following identities are derived

$$\begin{gathered} \mathcal{K}_{\alpha\beta}{}^{\beta} = iT_{\alpha}, \qquad \mathcal{K}_{\beta\gamma\alpha} I^{\gamma\beta} = J_{\alpha}, \qquad \mathcal{K}_{\beta\gamma\alpha}\Sigma_k^{\beta\gamma} = iT_{\beta}\Sigma_{k\,\alpha}^{\beta} \\ T^{\gamma}\mathcal{K}_{\gamma\beta\alpha} = -\frac{i}{2}(T_{\alpha}T_{\beta} - J_{\alpha}J_{\beta} - \mu^2\eta_{\alpha\beta}), \qquad J^{\gamma}\mathcal{K}_{\gamma\beta\alpha} = 0, \qquad \mathcal{K}_{\alpha\gamma\delta}\,\mathcal{K}_{\beta}{}^{\delta\gamma} = -\frac{1}{2}T_{\alpha}T_{\beta} \end{gathered} \tag{6.17}$$

which may be used to simplify (6.16) as follows:

$$R_{\alpha\beta} = \mathcal{R}_{\alpha\beta} - \frac{i}{2}(\mathrm{D}^{ch}_{\gamma}T_{\delta})(\eta^{\gamma}{}_{\beta}\eta^{\delta}{}_{\alpha} - I^{\gamma}{}_{\alpha}I^{\delta}{}_{\beta}) + \frac{1}{2}J_{\alpha}J_{\beta} + \frac{1}{2}(\mu^2 + \sigma)\eta_{\alpha\beta} \tag{6.18}$$

6.3 Einstein Connection for Spinors

In Dirac spinor notation the connection matrix of the Dirac connection (4.47) takes the form

$$\widehat{\Theta}_{\mathcal{D}} = \begin{pmatrix} \Theta_l & 0 \\ 0 & \Theta_r \end{pmatrix} = -i\mathrm{Re}\,\mathcal{A}\mathbf{1}_4 - \mathrm{Im}\,\mathcal{A}\gamma_5 + \frac{1}{2}\mathrm{Re}(L^k\Sigma^k_{\alpha\beta})S^{\alpha\beta} \tag{6.19}$$

as may be verified with help of (4.22) and (4.25). This matrix of 1-forms is expanded in a real Minkowski basis as follows

$$\widehat{\Theta}_{\mathcal{D}} = \Gamma^{\mathcal{D}}_{\alpha}\omega^{\alpha} + i\gamma_5\Gamma^{\mathcal{D}}_{\alpha}\tilde{\omega}^{\alpha}, \qquad \Gamma^{\mathcal{D}}_{\alpha} = -iA_{\alpha} - \widetilde{A}_{\alpha}\gamma_5 + \frac{1}{2}\mathrm{Re}(L^k_{\alpha}\Sigma^k_{\beta\gamma})S^{\beta\gamma} \tag{6.20}$$

where the duality properties of the Lorentz generators (4.16) and of spacetime Pauli matrices (4.22) have been used. The real and imaginary parts of the complex electromagnetic potential

$$A_{\alpha} = \mathrm{Re}\,\mathcal{A}_{\alpha}, \qquad \widetilde{A}_{\alpha} = \mathrm{Im}\,\mathcal{A}_{\alpha} \tag{6.21}$$

multiply the $\mathbf{U}(1)$ and $\mathbf{D}(1)$ generators and are called the *electromagnetic* and *axial vector potential* respectively. The complex electromagnetic field tensor correspondingly decomposes into the *electromagnetic* and *axial field tensors:*

$$F_{\alpha\beta} = \mathrm{Re}\,\mathcal{F}_{\alpha\beta} = 2\partial_{[\alpha}A_{\beta]} - A_{\gamma}\mathrm{Re}\,c^{\gamma}{}_{[\alpha\beta]}, \qquad \widetilde{F}_{\alpha\beta} = \mathrm{Im}\,\mathcal{F}_{\alpha\beta} = 2\partial_{[\alpha}\widetilde{A}_{\beta]} - \widetilde{A}_{\gamma}\mathrm{Re}\,c^{\gamma}{}_{[\alpha\beta]} \tag{6.22}$$

From (4.50) and (6.11) one may further derive the following expression for the complex electromagnetic potential in terms of the fundamental spinor field and the connection coefficients

$$\mathcal{A}_\alpha = \frac{1}{4}\left(\Gamma_{\beta\gamma\alpha} + \mathcal{K}_{\beta\gamma\alpha} + \frac{1}{2}\widetilde{\mathcal{C}}_{\beta\gamma\alpha}\right) I^{\gamma\beta} = -\frac{i}{2}\bar{\mu}^{-2}(\varphi^+ \overleftrightarrow{\partial}_\alpha \chi) + \frac{1}{4} J_\alpha + \frac{1}{4}\Gamma_{\beta\gamma\alpha} I^{\gamma\beta} \tag{6.23}$$

where the term involving $\widetilde{\mathcal{C}}$ has been evaluated with help of (4.42) using the chirality condition (4.3) in order to replace antichiral derivatives $\bar{\partial}^{ch}_\alpha$ with classical derivatives ∂_α.

In case of the Dirac connection the relation between the connection matrices in spinor and Minkowski bases was determined by the requirement (4.46) that the spinor tetrad fields be covariantly constant. The Einstein connection will instead be extended to spinors by defining the connection matrices in Dirac spinor notation as follows:

$$\widehat{\Theta}_{\mathcal{E}} = \Gamma^{\mathcal{E}}_\alpha \omega^\alpha + i\gamma_5 \Gamma^{\mathcal{E}}_\alpha \tilde{\omega}^\alpha, \qquad \Gamma^{\mathcal{E}}_\alpha = -i A_\alpha + \frac{1}{4}(\mathrm{Re}\,\Gamma_{\beta\gamma\alpha}) S^{\beta\gamma} \tag{6.24}$$

With this definition the tetrad fields are not covariantly constant with respect to the Einstein connection. Rather, the definition is chosen such that after symmetry breaking the Einstein connection for spinors becomes equal to the spin connection on spacetime.

In order to relate the spinor coefficients of both connections, the 1-forms L^k are expressed in terms of Δ using (4.50)

$$L^k_\alpha = \frac{1}{4}\left(\Delta_{\beta\gamma\alpha} - \frac{1}{2}\widetilde{\mathcal{C}}_{\beta\gamma\alpha}\right)\Sigma^{\gamma\beta}_k \tag{6.25}$$

where an additional term involving $\widetilde{\mathcal{C}}$ has been introduced, which vanishes according to (4.42). Replacing further the Dirac connection coefficients Δ with the Einstein connection coefficients Γ using (6.11), the Dirac spinor connection coefficients (6.20) take the form:

$$\Gamma^{\mathcal{D}}_\alpha = -i A_\alpha - \widetilde{A}_\alpha \gamma_5 + \frac{1}{8}\mathrm{Re}[(\Gamma_{\delta\epsilon\alpha} + \mathcal{K}_{\delta\epsilon\alpha})\Sigma^{\epsilon\delta}_k \Sigma^k_{\beta\gamma}]\, S^{\beta\gamma} \tag{6.26}$$

The term involving the contortion tensor may be simplified with help of (6.17). Using further the identity (4.23) and the duality property (4.16) of the Lorentz generators, (6.26) may be written as follows:

$$\Gamma^{\mathcal{D}}_\alpha = -i A_\alpha - \widetilde{A}_\alpha \gamma_5 + \frac{1}{4}[(\mathrm{Re} - i\gamma_5 \mathrm{Im})\Gamma_{\beta\gamma\alpha}]\, S^{\beta\gamma} - \frac{1}{4}[(\mathrm{Im} + i\gamma_5 \mathrm{Re}) T_\beta]\, S^\beta{}_\alpha \tag{6.27}$$

Subtracting the Einstein spinor connection coefficients one finally obtains:

$$K_\alpha = \Gamma^{\mathcal{D}}_\alpha - \Gamma^{\mathcal{E}}_\alpha = -\widetilde{A}_\alpha \gamma_5 - \frac{i}{4}\gamma_5(\mathrm{Im}\,\Gamma_{\beta\gamma\alpha}) S^{\beta\gamma} - \frac{1}{4}[(\mathrm{Im} + i\gamma_5 \mathrm{Re}) T_\beta] S^\beta{}_\alpha \tag{6.28}$$

6.4 Electromagnetic Einstein Equation

As a consequence of the contracted second Bianchi identities (5.33) the chiral covariant derivatives D^{ch}_α of the vector fields occurring on the right hand sides of the Einstein equations (5.39) and (5.40) may be replaced with covariant derivatives D_α in the directions of classical basis vectors e_α. The latter may further be expressed in terms of Einstein covariant derivatives ∇_α as follows

$$\mathrm{D}^{ch}_\alpha T_\beta = \mathrm{D}_\alpha T_\beta = \nabla_\alpha T_\beta + \frac{i}{2}(T_\alpha T_\beta - J_\alpha J_\beta - \mu^2 \eta_{\alpha\beta}), \qquad \mathrm{D}^{ch}_\alpha J_\beta = \mathrm{D}_\alpha J_\beta = \nabla_\alpha J_\beta \tag{6.29}$$

where the identities (6.17) have been used to evaluate the torsion terms. The electromagnetic Einstein equation (5.40) thus takes the form:

$$\mathcal{F}_{\alpha\beta} = \frac{1}{2}\nabla_{[\alpha} J_{\beta]} + \frac{i}{4}\epsilon_{\alpha\beta}{}^{\gamma\delta}\nabla_\gamma J_\delta - \frac{i}{2} J_{[\alpha} T_{\beta]} + \frac{1}{4}\mu^2 I_{\alpha\beta} \tag{6.30}$$

For products of vector fields the following identities hold

$$T_\alpha T_\beta - J_\alpha J_\beta = \frac{1}{2}\mu^2(\eta + I\bar{I})_{\alpha\beta}, \qquad J_{[\alpha} T_{\beta]} = \frac{1}{2}\mu^2 \mathrm{Im}\, I_{\alpha\beta} \tag{6.31}$$

which are derived with help of (4.43) and the Fiertz identities (4.17) as follows:

$$\begin{aligned}(\eta + I\bar{I} - iI + i\bar{I})^\alpha{}_\beta &= \xi^\alpha_C \xi^C_\gamma \xi^\gamma_{\dot{D}} \xi^{\dot{D}}_\beta = 2(\mu\bar{\mu})^{-2}(\varphi^+\sigma^\alpha\varphi)(\chi^+\hat{\sigma}_\beta\chi)\\ &= 2\mu^{-2}(T^\alpha + J^\alpha)(T_\beta - J_\beta)\end{aligned} \tag{6.32}$$

The relations (6.31) are obtained as the symmetric and antisymmetric part respectively of (6.32), taking into account that $(I\bar{I})_{\alpha\beta}$ is symmetric according to (4.22).

Using the second of these identities, the right hand side of (6.30) may be simplified further and the electromagnetic Einstein equation finally takes the form:

$$\mathcal{F}_{\alpha\beta} = \frac{1}{4}\mu^2 \mathrm{Re}\, I_{\alpha\beta} + \frac{1}{2}\nabla_{[\alpha} J_{\beta]} + \frac{i}{4}\epsilon_{\alpha\beta}{}^{\gamma\delta}\nabla_\gamma J_\delta \tag{6.33}$$

6.5 Gravitational Einstein Equation

In the gravitational Einstein equation (5.39) the Dirac-Einstein tensor is replaced with the Einstein tensor of the Einstein connection using the relation (6.18):

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} R^\gamma{}_\gamma = -i\mathrm{D}_{(\alpha} T_{\beta)} + \frac{1}{2} T_\alpha T_\beta - \frac{1}{2}\left(\sigma + \frac{1}{2}\mu^2\right)\eta_{\alpha\beta} \tag{6.34}$$

The covariant derivative of the torsion vector may be expressed in terms of derivatives of the fundamental spinor field as follows

$$\mathrm{D}_\alpha T_\beta = \frac{1}{2}\bar{\mu}^{-1}[\varphi^+\sigma_\beta \mathrm{D}_\alpha\varphi + (\mathrm{D}_\alpha\chi^+)\hat{\sigma}_\beta\chi] \tag{6.35}$$

where the covariant constancy of the spinor tetrad fields has been used. Introducing a Dirac spinor notation for the fundamental spinor field

$$\Psi = \begin{pmatrix}\chi\\ \varphi\end{pmatrix}, \qquad \bar{\Psi} = \Psi^+\beta; \qquad T_\alpha = \frac{i}{2}\bar{\mu}^{-1}\bar{\Psi}\gamma_5\gamma_\alpha\Psi, \qquad J_\alpha = \frac{i}{2}\bar{\mu}^{-1}\bar{\Psi}\gamma_\alpha\Psi \tag{6.36}$$

and replacing the Dirac covariant derivatives with Einstein covariant derivatives, (6.35) may be evaluated further as follows

$$\begin{aligned}\mathrm{D}_\alpha T_\beta &= \frac{i}{2}\bar{\mu}^{-1}[\bar{\Psi}\gamma_\beta P_r \mathrm{D}_\alpha\Psi - (\mathrm{D}_\alpha\bar{\Psi})\gamma_\beta P_l\Psi]\\ &= \frac{i}{4}\bar{\mu}^{-1}[\bar{\Psi}\overset{\leftrightarrow}{\nabla}_\alpha\gamma_\beta\Psi + \nabla_\alpha(\bar{\Psi}\gamma_5\gamma_\beta\Psi)] + \frac{i}{2}\bar{\mu}^{-1}[\bar{\Psi}\gamma_\beta P_r K_\alpha\Psi - \bar{\Psi}\beta K^+_\alpha\beta P_r\gamma_\beta\Psi]\\ &\quad + \frac{i}{2}(T_\gamma + J_\gamma)\mathrm{Im}\,\Gamma^\gamma{}_{\beta\alpha}\end{aligned} \tag{6.37}$$

with chiral projectors (4.6). The last term compensates for the derivatives of the Dirac matrices γ_β introduced in the first two terms of the second line. In general, the covariant derivative of a Dirac matrix is not well-defined in spinor relativity. In the present case the Lorentz index of γ_β has its origin in the vector field T_β and thus refers to the bundle $U^*(M_{ch})$, which determines its covariant derivative as follows:

$$\nabla_\alpha \gamma_\beta \Psi = \gamma_\beta \partial_\alpha \Psi + \Gamma^{\mathcal{E}}_\alpha \gamma_\beta \Psi - \gamma_\delta \Psi \Gamma^\delta{}_{\beta\alpha} = \gamma_\beta \nabla_\alpha \Psi - i\gamma_\delta \Psi \mathrm{Im}\,\Gamma^\delta{}_{\beta\alpha} \tag{6.38}$$

The terms involving K may be evaluated with help of (6.28) and the (anti-)commutation relations (4.15) between Lorentz generators and Dirac matrices

$$\begin{aligned}
&P_r K_\alpha = -\beta K_\alpha^+ \beta P_r = P_r \left[\widetilde{A}_\alpha + \frac{i}{4}(T_\gamma \eta_{\beta\alpha} + \mathrm{Im}\,\Gamma_{\gamma\beta\alpha}) S^{\gamma\beta}\right] \\
&\frac{i}{2}\bar{\mu}^{-1}[\bar{\Psi}\gamma_\beta P_r K_\alpha \Psi - \bar{\Psi}\beta K_\alpha^+ \beta P_r \gamma_\beta \Psi] \\
&\quad = \widetilde{A}_\alpha J_\beta + \frac{i}{2}(T_{[\gamma}\eta_{\delta]\alpha} + \mathrm{Im}\,\Gamma_{\gamma\delta\alpha})\left(\eta_\beta{}^\gamma T^\delta + \frac{i}{2}\epsilon_{\beta\epsilon}{}^{\gamma\delta} T^\epsilon\right) \\
&\quad = \widetilde{A}_\alpha J_\beta + \frac{i}{4}(T_\alpha T_\beta - \mu^2 \eta_{\alpha\beta}) + \frac{1}{2} T^\gamma (\mathrm{Im}\,\Gamma^*_{\gamma\beta\alpha} - i\,\mathrm{Im}\,\Gamma_{\gamma\beta\alpha})
\end{aligned} \tag{6.39}$$

where the asterisk denotes the dual with respect to its first index pair of the connection coefficient. The Dirac covariant derivative of the torsion vector may thus be written as follows:

$$\begin{aligned}
\mathrm{D}_\alpha T_\beta = {}& \frac{i}{4}\bar{\mu}^{-1}(\bar{\Psi}\overset{\leftrightarrow}{\nabla}_\alpha \gamma_\beta \Psi) + \frac{1}{2}(\nabla_\alpha T_\beta + T_\beta \partial_\alpha \ln \bar{\mu}) \\
&+ \widetilde{A}_\alpha J_\beta + \frac{i}{4}(T_\alpha T_\beta - \mu^2 \eta_{\alpha\beta}) + \frac{1}{2}(T^\gamma \mathrm{Im}\,\Gamma^*_{\gamma\beta\alpha} + i J^\gamma \mathrm{Im}\,\Gamma_{\gamma\beta\alpha})
\end{aligned} \tag{6.40}$$

Comparison with (6.29) allows to eliminate the Einstein derivative of the torsion vector, and the desired expression of the Dirac derivative of the torsion vector in terms of Einstein derivatives of the fundamental spinor field is given by:

$$\mathrm{D}_\alpha T_\beta = \frac{i}{2}\bar{\mu}^{-1}(\bar{\Psi}\overset{\leftrightarrow}{\nabla}_\alpha \gamma_\beta \Psi) + \frac{i}{2} J_\alpha J_\beta + 2\widetilde{A}_\alpha J_\beta + T_\beta \partial_\alpha \ln \bar{\mu} + T^\gamma \mathrm{Im}\,\Gamma^*_{\gamma\beta\alpha} + i J^\gamma \mathrm{Im}\,\Gamma_{\gamma\beta\alpha} \tag{6.41}$$

Inserting (6.41) into (6.34) and eliminating the product of two current vectors with help of (6.31), the gravitational Einstein equation finally takes the form:

$$\begin{aligned}
G_{\alpha\beta} = {}& \frac{1}{2}\bar{\mu}^{-1}[\bar{\Psi}\overset{\leftrightarrow}{\nabla}_{(\alpha}\gamma_{\beta)}\Psi] + T_\alpha T_\beta - \frac{1}{4}\mu^2 (I\bar{I})_{\alpha\beta} - \frac{1}{2}(\sigma + \mu^2)\eta_{\alpha\beta} \\
&- 2i\widetilde{A}_{(\alpha} J_{\beta)} - i T_{(\beta}\partial_{\alpha)} \ln \bar{\mu} - i T^\gamma \mathrm{Im}\,\Gamma^*_{\gamma(\beta\alpha)} + J^\gamma \mathrm{Im}\,\Gamma_{\gamma(\beta\alpha)}
\end{aligned} \tag{6.42}$$

6.6 Maxwell Equations

The homogeneous Maxwell equation is obtained from the contracted first Bianchi identity for the dual electromagnetic field tensor (5.49) by replacing the Dirac divergence with the Einstein divergence

$$\mathrm{D}_\beta \mathcal{F}^{*\alpha\beta} = \nabla_\beta \mathcal{F}^{*\alpha\beta} - i\mathcal{F}^{*\alpha\beta} T_\beta \tag{6.43}$$

where it has been taken into account that $\mathcal{F}^*$ commutes with the complex spinor structure.[13] The term involving the torsion vector cancels the corresponding term in the Bianchi identity and the *homogeneous Maxwell equation* takes the form:[14]

$$\nabla_\beta \mathcal{F}^{*\alpha\beta} = 0 \tag{6.44}$$

The inhomogeneous Maxwell equation is derived from the divergence of (5.21) and the homogeneous equation (6.45)

$$0 = \nabla_\beta \mathcal{F}^{\alpha\beta} - \frac{1}{4}\nabla_\beta(\mu^2 I^{\alpha\beta}) \tag{6.45}$$

where the curvature scalar ρ has been replaced by its value (5.34) from the dynamical equations. Inserting the Einstein divergence of the complex spinor structure

$$\nabla_\beta I^{\alpha\beta} = -J^\alpha \tag{6.46}$$

the *inhomogeneous Maxwell equation* takes the form:

$$\nabla_\beta \mathcal{F}^{\alpha\beta} = -\frac{1}{4}\mu^2(J^\alpha - I^{\alpha\beta}\partial_\beta \ln \mu^2) \tag{6.47}$$

6.7 Dirac Identity

From the covariant constancy of the spinor tetrad fields with respect to the Dirac connection the following expressions for their Einstein covariant derivatives are obtained

$$\nabla_\alpha \xi^A_\beta + (K^r_\alpha)^A{}_B \xi^B_\beta - \xi^A_\gamma \mathcal{K}^\gamma{}_{\beta\alpha} = 0, \qquad \nabla_\alpha \zeta^A_\beta + (K^l_\alpha)^A{}_B \zeta^B_\beta - \zeta^A_\gamma \bar{\mathcal{K}}^\gamma{}_{\beta\alpha} = 0 \tag{6.48}$$

where K^l and K^r denote the upper and lower parts respectively of the block diagonal matrix K defined in (6.28). On the other hand, the tetrad fields constitute up to scalar factors the Dirac spinor $\gamma_\beta\Psi$:

$$i\gamma_\beta\Psi = \begin{pmatrix} \sigma_\beta\varphi \\ -\hat{\sigma}_\beta\chi \end{pmatrix} = \begin{pmatrix} \mu\,\zeta^A_\beta \\ -\bar{\mu}\,\xi^A_\beta \end{pmatrix} \tag{6.49}$$

The two equations (6.48) may thus be combined into a single equation as follows:

$$\nabla_\alpha\gamma_\beta\Psi - [(\mathrm{Re} + i\gamma_5\mathrm{Im})\partial_\alpha \ln\mu]\gamma_\beta\Psi + K_\alpha\gamma_\beta\Psi - [(\mathrm{Re} - i\gamma_5\mathrm{Im})\mathcal{K}^\delta{}_{\beta\alpha}]\gamma_\delta\Psi = 0 \tag{6.50}$$

In evaluating the covariant derivative of γ_β it must be taken into account that the Lorentz index in the upper and lower part of (6.49) refers to the bundles $\bar{U}^*(M_{ch})$ and $U^*(M_{ch})$

[13] In general, the relation between the Dirac and Einstein divergences of an arbitrary tensor field $M^{\alpha\beta}$ is given by:

$$\mathrm{D}_\beta M^{\alpha\beta} = \nabla_\beta M^{\alpha\beta} - 2iM^{(\alpha\beta)}T_\beta - iM^{[\alpha\beta]}T_\beta + \frac{1}{2}[M,I]^{\alpha\beta}J_\beta + \frac{i}{2}T^\alpha M^\beta{}_\beta.$$

[14] From the contracted first Bianchi identity for $\mathcal{G}$ one obtains the corresponding identity $\nabla_\beta G^\beta{}_\alpha = 0$ for the Einstein tensor of the Einstein connection, which of course also follows immediately from the first Bianchi identity for the torsion-free Einstein connection. This equation will however not be considered explicitly in the following.

respectively:

$$\nabla_\alpha \gamma_\beta \Psi = \gamma_\beta \partial_\alpha \Psi + \Gamma^{\varepsilon}_{\alpha} \gamma_\beta \Psi + i \begin{pmatrix} \sigma_\gamma \varphi \, \bar{\Gamma}^{\gamma}{}_{\beta\alpha} \\ -\hat{\sigma}_\gamma \chi \, \Gamma^{\gamma}{}_{\beta\alpha} \end{pmatrix} = \gamma_\beta \nabla_\alpha \Psi + i \gamma_5 \gamma_\delta \Psi \mathrm{Im}\, \Gamma^{\delta}{}_{\beta\alpha} \tag{6.51}$$

This result is inserted into (6.50) and the Lorentz indices are contracted with the Minkowski metric. The third term may be evaluated with help of (6.28):

$$K_\alpha \gamma^\alpha = -\widetilde{A}_\alpha \gamma_5 \gamma^\alpha - \frac{1}{2}(\mathrm{Im}\, \Gamma^{*}_{\alpha\beta}{}^{\beta} + i\gamma_5 \mathrm{Im}\, \Gamma_{\alpha\beta}{}^{\beta})\gamma^\alpha - \frac{3}{4}[(\mathrm{Im} + i\gamma_5 \mathrm{Re})T_\alpha]\gamma^\alpha \tag{6.52}$$

From contracting the contortion tensor in (6.50) a term of the same form as the last term on the right hand side of (6.52) arises, but with a factor of unity. Finally, one obtains an identity for the fundamental spinor field, which is independent of the dynamical equations of spinor relativity and is called the *Dirac identity:*

$$\begin{aligned} \gamma^\alpha \nabla_\alpha \Psi = \bigg[&(\mathrm{Re} + i\gamma_5 \mathrm{Im})\partial_\alpha \ln \mu + \widetilde{A}_\alpha \gamma_5 + \frac{1}{2}(\mathrm{Im}\, \Gamma^{*}_{\alpha\beta}{}^{\beta} - i\gamma_5 \mathrm{Im}\, \Gamma_{\alpha\beta}{}^{\beta}) \\ &- \frac{1}{4}(\mathrm{Im} + i\gamma_5 \mathrm{Re})T_\alpha \bigg] \gamma^\alpha \Psi \end{aligned} \tag{6.53}$$

7 Symmetry Breaking

The Dirac manifold is considered as a quantum manifold in the sense that the Dirac metric and the fields derived from it are operator valued. In contrast to quantum gravity, where it is usually assumed that the metric has a large classical part, which is responsible for the classical appearance of spacetime, the Dirac metric is supposed to be far from classical behaviour and the eight-dimensional manifold may not be imagined as a classical manifold. In this chapter it is shown how spacetime might be understood as arising from a degenerate ground state of spinor relativity, which breaks $\mathbf{D}(1)$-gauge symmetry as well as covariance with respect to independent coordinate transformations on the right and left manifolds. The development of the ground state involves the choice of a preferred four-dimensional submanifold $\mathcal{S}$ of the Dirac manifold, which attains classical properties and is equipped with a Lorentz metric.

Consistency requires the energy scale set by the degenerate ground state to be of Planck order, which means that fields at ordinary energy scales may be considered as small excitations above this ground state. The usual Einstein and Maxwell equations describing real electromagnetic and gravitational fields coupled to a spinor field may be derived from the Heisenberg equation system as an approximation valid in the vicinity of the ground state, where the Planck length and the fine structure constant are related to the scale of the ground state. The electromagnetic Einstein equation further leads to an interpretation of the electromagnetic field as arising from the polarization of the ground state condensate.

7.1 The Ground State

The ground state expectation value of the invariant square of the fundamental spinor field is assumed to be positive:

$$\langle \chi^{+} \varphi \rangle_0 = \mu_0^2 > 0 \quad \Leftrightarrow \quad \langle \bar{\Psi}\Psi \rangle_0 = 2\mu_0^2, \qquad \langle \bar{\Psi}\gamma_5 \Psi \rangle_0 = 0 \tag{7.1}$$

There is further the possibility for products of right or left spinor fields with dyad fields to have Lorentz invariant ground state expectation values, which in spinor relativity are supposed to take the form:

$$\langle \varphi^A \varphi_B^+ E_r^C{}_M E_l^M{}_D \rangle_0 = \frac{1}{2}\mu_0^2 \epsilon_r^{AC} \epsilon_{BD}^l, \qquad \langle \chi^A \chi_B^+ E_l^C{}_M E_r^M{}_D \rangle_0 = \frac{1}{2}\mu_0^2 \epsilon_l^{AC} \epsilon_{BD}^r \tag{7.2}$$

The matrices on the right hand sides are $\mathbf{SL}(\mathbb{C}^2)$ and $\mathbf{U}(1)$ invariant, but are multiplied by dilatation factors on $\mathbf{D}(1)$-transformations, which means that $\mathbf{D}(1)$-gauge symmetry is broken. Since the chiral coordinate indices on the right and left dyad fields in (7.2) refer to the right and left manifold respectively, their contraction in products of dyad fields further breaks the chiral symmetry of independent coordinate transformations on the right and left manifolds.

These ground state expectation values allow to define a four-dimensional submanifold $\mathcal{S}$ of the Dirac manifold as consisting of pairs of points on the right and left manifold, which satisfy the condition

$$z_r^M|_{\mathcal{S}} = z_l^M|_{\mathcal{S}} = z^M \tag{7.3}$$

with respect to chiral coordinates compatible with (7.2). This condition is left invariant by the unbroken symmetry of joint coordinate transformations on both manifolds. The ground state thus distinguishes a preferred submanifold $\mathcal{S}$, which will be identified with *spacetime*. In this chapter attention is restricted to spacetime and the following results are valid on $\mathcal{S}$ only. In particular, the possibility is left open that the ground state expectation values may vary on the Dirac manifold such that the functions, which replace μ_0^2 on the right hand sides of (7.1) and (7.2) respectively, take equal constant values on $\mathcal{S}$ but vary independently on leaving the spacetime submanifold.

In restricting attention to functions on spacetime, it has to be taken into account that the differential further distinguishes between right and left coordinates, which may be identified only after derivatives have been performed. This differential structure on $\mathcal{S}$ may be alternatively described without reference to the right and left manifold as follows. Since in spinor relativity all fields derive from the dyad fields, which satisfy the chirality conditions (3.21), an arbitrary field ϕ on $\mathcal{S}$ may be written as a sum of products of chiral and antichiral fields:

$$\phi(z) = \sum_\nu \phi_\nu^+(z)\phi_\nu^-(z), \qquad \bar{d}_{ch}\phi_\nu^+ = 0, \qquad d_{ch}\phi_\nu^- = 0 \tag{7.4}$$

Instead of right and left coordinate derivatives it is convenient to introduce the following linear combinations:

$$\partial_M = \partial_M^r + \partial_M^l, \qquad \tilde{\partial}_M = \partial_M^r - \partial_M^l \tag{7.5}$$

∂_M is the classical derivative on $\mathcal{S}$ with respect to the joint coordinate z^M, which does not distinguish between chiral and antichiral fields. The axial derivative $\tilde{\partial}_M$ on the other hand acts on antichiral fields with a negative sign:

$$\partial_M\phi(z) = \frac{\partial}{\partial z^M}\phi(z), \qquad \tilde{\partial}_M\phi(z) = \sum_\nu \left[\phi_\nu^-(z)\frac{\partial}{\partial z^M}\phi_\nu^+(z) - \phi_\nu^+(z)\frac{\partial}{\partial z^M}\phi_\nu^-(z)\right] \tag{7.6}$$

It will be seen that μ_0 is of Planck order, which means that fields at usual energies may be considered as small excitations above the ground state. One may expect to obtain a good approximation for the description of these excitations, if the expressions on the left hand sides of (7.1) and (7.2) are replaced with their ground state expectation values. This approximation is called the *vacuum approximation* and will be applied to the Heisenberg equation system in the next section. It is now shown that spacetime becomes equipped with a Lorentz metric in the vacuum approximation.

To this end a field Z is introduced, which transforms right and left spinor tetrad fields into each other:

$$\xi_\beta^M Z^\beta{}_\alpha = \zeta_\alpha^M, \qquad \zeta_\beta^{\dot M} Z^\beta{}_\alpha = \xi_\alpha^{\dot M}; \qquad Z^\alpha{}_\beta \zeta_M^\beta = \xi_M^\alpha, \qquad Z^\alpha{}_\beta \xi_{\dot M}^\beta = \zeta_{\dot M}^\alpha \tag{7.7}$$

As may be seen from the identities (4.28) this is accomplished by the following choice:

$$Z^\alpha{}_\beta = \frac{1}{2}(\xi_M^\alpha \zeta_\beta^M + \zeta_{\dot M}^\alpha \xi_\beta^{\dot M}) \tag{7.8}$$

Contracting (7.2) with Weyl matrices and taking into account the duality properties (4.18), one obtains the following ground state expectation values for products of spinor tetrad fields

$$\begin{aligned} \langle \xi_M^\alpha \zeta_\beta^M \rangle_0 &= \frac{1}{2}(\sigma^\alpha)^B{}_C (\sigma_\beta)^D{}_A \epsilon_r^{AC} \epsilon_{BD}^l = \eta^\alpha{}_\beta, \\ \langle \zeta_M^\alpha \xi_\beta^M \rangle_0 &= \frac{1}{2}(\hat\sigma^\alpha)^B{}_C (\hat\sigma_\beta)^D{}_A \epsilon_l^{AC} \epsilon_{BD}^r = \eta^\alpha{}_\beta \end{aligned} \tag{7.9}$$

which show that Z has unit ground state expectation value:

$$\langle Z^\alpha{}_\beta \rangle_0 = \eta^\alpha{}_\beta \tag{7.10}$$

The choice of $\mathbf{D}(1)$-gauge enforced by (7.2) implies an invariant decomposition of the tangent spaces of the Dirac manifold into classical and axial vectors, which do not mix under transformations from the unbroken gauge group $\mathbf{SL}(\mathbb{C}^2) \circ \mathbf{U}(1)$. In the vacuum approximation the classical vectors e_α may be identified with the tangent vectors of $\mathcal{S}$. In order to see this the derivatives ∂_α and $\tilde\partial_\alpha$ in the directions of classical and axial basis vectors e_α and $\tilde e_\alpha$ respectively are considered, which may be written in terms of Z and the derivatives (7.5) as follows:

$$\begin{aligned} \partial_\alpha &= \mathrm{Re}(\xi_\alpha^M \partial_M^r + \zeta_\alpha^M \partial_M^l) = \frac{1}{2}\mathrm{Re}[(\eta + Z)^\beta{}_\alpha \xi_\beta^M \partial_M + (\eta - Z)^\beta{}_\alpha \xi_\beta^M \tilde\partial_M] \\ \tilde\partial_\alpha &= \mathrm{Im}(\xi_\alpha^M \partial_M^r - \zeta_\alpha^M \partial_M^l) = \frac{1}{2}\mathrm{Im}[(\eta - Z)^\beta{}_\alpha \xi_\beta^M \partial_M + (\eta + Z)^\beta{}_\alpha \xi_\beta^M \tilde\partial_M] \end{aligned} \tag{7.11}$$

In the vacuum approximation Z is replaced with its ground state expectation value (7.10), yielding the simplified expressions

$$\partial_\alpha \overset{\circ}{=} \mathrm{Re}(\xi_\alpha^M \partial_M), \qquad \tilde\partial_\alpha \overset{\circ}{=} \mathrm{Im}(\xi_\alpha^M \tilde\partial_M) \tag{7.12}$$

where $\overset{\circ}{=}$ denotes equality within the vacuum approximation. It is seen that ∂_α is a combination of classical derivatives ∂_M on spacetime and does not involve contributions from the axial derivatives $\tilde\partial_M$. This means that the spaces spanned by the basis vectors e_α are equal to the tangent spaces of $\mathcal{S}$ within the vacuum approximation. Since further the Dirac product is given by the Minkowski metric (6.3) on classical basis vectors, spacetime is equipped with a Lorentz metric, with respect to which the vectors e_α constitute an orthonormal tetrad basis.

These results are confirmed considering the structure functions (6.5) of real Minkowski bases. Using Z, the real and imaginary parts of c may be written as follows:

$$\begin{aligned}\operatorname{Re} c^{\alpha}{}_{\beta\gamma} &= \operatorname{Re}[(Z^{\alpha}{}_{\epsilon}\eta^{\delta}{}_{\beta} + \eta^{\alpha}{}_{\epsilon}Z^{\delta}{}_{\beta})\xi^{M}_{\delta}\partial_{\gamma}\zeta^{\epsilon}_{M} + (\partial_{\gamma}Z^{\alpha}{}_{\epsilon})\xi^{M}_{\beta}\zeta^{\epsilon}_{M}] \\ \operatorname{Im} c^{\alpha}{}_{\beta\gamma} &= \operatorname{Im}[(Z^{\alpha}{}_{\epsilon}\eta^{\delta}{}_{\beta} - \eta^{\alpha}{}_{\epsilon}Z^{\delta}{}_{\beta})\xi^{M}_{\delta}\partial_{\gamma}\zeta^{\epsilon}_{M} + (\partial_{\gamma}Z^{\alpha}{}_{\epsilon})\xi^{M}_{\beta}\zeta^{\epsilon}_{M}]\end{aligned} \tag{7.13}$$

Similar expressions are obtained for the real and imaginary parts of $\tilde{c}$, involving the axial derivatives $\tilde{\partial}_{\gamma}$ instead of ∂_{γ}. Replacing again Z with its ground state expectation value one obtains the approximate expressions

$$\begin{aligned}\operatorname{Re} c^{\alpha}{}_{\beta\gamma} &\mathrel{\overset{\circ}{=}} 2\operatorname{Re}(\xi^{M}_{\beta}\partial_{\gamma}\zeta^{\alpha}_{M}), \qquad \operatorname{Im} c^{\alpha}{}_{\beta\gamma} \mathrel{\overset{\circ}{=}} 0, \\ \operatorname{Re} \tilde{c}^{\alpha}{}_{\beta\gamma} &\mathrel{\overset{\circ}{=}} 2\operatorname{Re}(\xi^{M}_{\beta}\tilde{\partial}_{\gamma}\zeta^{\alpha}_{M}), \qquad \operatorname{Im} \tilde{c}^{\alpha}{}_{\beta\gamma} \mathrel{\overset{\circ}{=}} 0\end{aligned} \tag{7.14}$$

and the commutation relations (6.6) of basis vectors simplify as follows:

$$[e_{\alpha}, e_{\beta}] \mathrel{\overset{\circ}{=}} c^{\gamma}{}_{[\alpha\beta]}e_{\gamma}, \qquad [\tilde{e}_{\alpha}, \tilde{e}_{\beta}] \mathrel{\overset{\circ}{=}} \tilde{c}^{\gamma}{}_{[\alpha\beta]}\tilde{e}_{\gamma}, \qquad [\tilde{e}_{\alpha}, e_{\beta}] \mathrel{\overset{\circ}{=}} \frac{1}{2}(c^{\gamma}{}_{\alpha\beta}\tilde{e}_{\gamma} - \tilde{c}^{\gamma}{}_{\beta\alpha}e_{\gamma}) \tag{7.15}$$

According to Frobenius's theorem this shows in particular that the subspaces of the eight-dimensional tangent spaces spanned by the classical vectors e_{α} are integrable in the vacuum approximation, forming the tangent bundle of a submanifold, which according to the previous results is spacetime.

In order that spacetime appear as a classical four-dimensional manifold it is further necessary that the commutators of its tangent basis vectors e_{α} have only small quantum fluctuations, while commutators involving axial basis vectors $\tilde{e}_{\alpha}$ fluctuate strongly. In this case the metric on $\mathcal{S}$ is essentially a classical metric, but in tangent directions leaving $\mathcal{S}$ there is no unique notion of distance between points and the manifold has no classical appearance.

7.2 Heisenberg Equation System for Broken Symmetry

In this section the vacuum approximation is applied to the Heisenberg equation system: The square of the fundamental spinor field appearing on the right hand sides of (6.33), (6.42), (6.47) and (6.53) is set equal to its ground state expectation value (7.1):

$$\mu \mathrel{\overset{\circ}{=}} \mu_0 \tag{7.16}$$

According to (6.36) this implies that the torsion and current vectors become real:

$$\operatorname{Im} T_{\alpha} \mathrel{\overset{\circ}{=}} 0, \qquad \operatorname{Im} J_{\alpha} \mathrel{\overset{\circ}{=}} 0 \tag{7.17}$$

In the last section it has been shown that in the vacuum approximation the structure functions c are real. From (6.12) then follows that the same is true for the connection coefficients Γ

$$\operatorname{Im} \Gamma^{\alpha}{}_{\beta\gamma} \mathrel{\overset{\circ}{=}} 0 \tag{7.18}$$

and the Einstein connection, restricted to derivatives in tangent directions on $\mathcal{S}$, becomes equal to the Levi-Civita connection of the Lorentz metric on spacetime.

Applying these approximations to the Maxwell equations (6.44) and (6.47), they decompose into separate equations for the electromagnetic and axial field tensors

$$\nabla_{\beta}F^{\alpha\beta} \mathrel{\overset{\circ}{=}} -\frac{1}{8}\mu_0 j^{\alpha}, \qquad \nabla_{\beta}F^{*\alpha\beta} \mathrel{\overset{\circ}{=}} 0; \qquad \nabla_{\beta}\widetilde{F}^{\alpha\beta} \mathrel{\overset{\circ}{=}} 0, \qquad \nabla_{\beta}\widetilde{F}^{*\alpha\beta} \mathrel{\overset{\circ}{=}} 0 \tag{7.19}$$

with Dirac current given by:

$$j^\alpha = i\bar{\Psi}\gamma^\alpha\Psi \tag{7.20}$$

Although its source vanishes, the axial field itself does not completely vanish, as is shown by the electromagnetic Einstein equation (6.33), which in the vacuum approximation decomposes as follows:

$$F_{\alpha\beta} \mathrel{\stackrel{\circ}{=}} \frac{1}{4}\mu_0^2 \mathrm{Re}\, I_{\alpha\beta} + \frac{1}{4}\mu_0^{-1}\nabla_{[\alpha} j_{\beta]}, \qquad \widetilde{F}_{\alpha\beta} \mathrel{\stackrel{\circ}{=}} \frac{1}{8}\mu_0^{-1}\epsilon_{\alpha\beta}{}^{\gamma\delta}\nabla_\gamma j_\delta \tag{7.21}$$

The axial field is proportional to the dual of the rotation of the Dirac current and is thus restricted to the interior of matter. Moreover, it will be seen that it is suppressed by a factor of λ_P^2. A similar small term also appears in the Einstein equation for the electromagnetic field, the main part of which however arises from the *polarization tensor*:

$$F_{\alpha\beta} + \widetilde{F}^*_{\alpha\beta} = P_{\alpha\beta} \mathrel{\stackrel{\circ}{=}} \frac{1}{4}\mu_0^2 \mathrm{Re}\, I_{\alpha\beta} \tag{7.22}$$

Using the identity

$$\frac{i}{2}\bar{\Psi} S_{\alpha\beta}\Psi = \mathrm{Re}(\varphi^+ \chi I_{\alpha\beta}) \tag{7.23}$$

which is derived with help of (4.25) and (4.29), it is seen that the polarization tensor may be written in the form:

$$P_{\alpha\beta} \mathrel{\stackrel{\circ}{=}} \frac{i}{16}\bar{\Psi}[\gamma_\alpha, \gamma_\beta]\Psi \tag{7.24}$$

The special form of the electromagnetic field provided by the electromagnetic Einstein equation yields a simple expression for its energy-momentum tensor. In order to derive it, the electromagnetic and axial field vectors are introduced

$$E_k = F_{k0}, \qquad B_k = \frac{1}{2}\epsilon_{kmn} F_{mn}; \qquad H_k = \widetilde{F}_{k0}, \qquad D_k = -\frac{1}{2}\epsilon_{kmn}\widetilde{F}_{mn} \tag{7.25}$$

in terms of which (7.22) takes the form:

$$D_k - E_k \mathrel{\stackrel{\circ}{=}} \frac{1}{4}\mu_0^2\, \mathrm{Im}\, n_k, \qquad B_k - H_k \mathrel{\stackrel{\circ}{=}} -\frac{1}{4}\mu_0^2\, \mathrm{Re}\, n_k \tag{7.26}$$

On the other hand, the energy-momentum tensor of the electromagnetic field F may be written in terms of spacetime Pauli matrices as follows:

$$T^{em}_{\alpha\beta} = F_{\alpha\gamma} F_\beta{}^\gamma - \frac{1}{4}\eta_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} = -\frac{1}{2}(E_m - iB_m)(E_n + iB_n)(\bar{\Sigma}^m \Sigma^n)_{\alpha\beta} \tag{7.27}$$

Inserting the relations (7.26) with axial fields neglected, it is seen with help of (4.29) that the energy-momentum tensor of the electromagnetic field in the vacuum approximation may be expressed in terms of the complex spinor structure as follows:

$$T^{em}_{\alpha\beta} \mathrel{\stackrel{\circ}{=}} -\frac{1}{32}\mu_0^4\, (I\bar{I})_{\alpha\beta} \tag{7.28}$$

The vacuum approximation of the Dirac identity (6.53) takes the form

$$\gamma^\alpha\nabla_\alpha\Psi - \frac{1}{8}\mu_0^{-1}(\bar{\Psi}\gamma_5\gamma_\alpha\Psi)\gamma_5\gamma^\alpha\Psi \mathrel{\stackrel{\circ}{=}} 0 \tag{7.29}$$

where the second term on the right hand side of (6.53) involving the axial vector potential has been neglected due to its smallness. As a consequence of the Fiertz identities the trilinear interaction term may be written in several equivalent forms:

$$-(\bar{\Psi}\gamma_5\gamma_\alpha\Psi)\gamma_5\gamma^\alpha\Psi = (\bar{\Psi}\gamma_\alpha\Psi)\gamma^\alpha\Psi = (\bar{\Psi}\Psi)\Psi - (\bar{\Psi}\gamma_5\Psi)\gamma_5\Psi \tag{7.30}$$

In applying the vacuum approximation to the gravitational Einstein equation (6.42), the possibility is taken into account that the Einstein tensor receives a ground state contribution from terms in (6.15) quadratic in the imaginary part of the connection coefficients

$$G_{\alpha\beta} \overset{\circ}{=} G^{cl}_{\alpha\beta} + \Lambda^G \eta_{\alpha\beta}, \qquad \Lambda^G = \frac{1}{4}\langle \operatorname{Im}\Gamma_{\alpha\beta}{}^{\beta}\operatorname{Im}\Gamma^{\alpha\gamma}{}_{\gamma} - \operatorname{Im}\Gamma_{\alpha\beta\gamma}\operatorname{Im}\Gamma^{\alpha\gamma\beta}\rangle_0 \tag{7.31}$$

where G^{cl} denotes the classical Einstein tensor arising from the real part of the connection coefficients. Neglecting the term involving the axial vector potential on the right hand side of (6.42), the Einstein equation takes the approximate form:

$$G^{cl}_{\alpha\beta} \overset{\circ}{=} \frac{1}{2}\mu_0^{-1}\bar{\Psi}\overset{\leftrightarrow}{\nabla}_{(\alpha}\gamma_{\beta)}\Psi + T_\alpha T_\beta - \frac{1}{4}\mu_0^2(I\bar{I})_{\alpha\beta} - \left[\Lambda^G + \frac{1}{2}(\sigma + \mu_0^2)\right]\eta_{\alpha\beta} \tag{7.32}$$

Comparison with (7.28) shows that the third term on the right hand side is proportional to the energy-momentum tensor of the electromagnetic field. The ground state approximation of σ is obtained from the trace of (6.41) as follows

$$\sigma \overset{\circ}{=} \frac{1}{2}\mu_0^{-1}\bar{\Psi}\overset{\leftrightarrow}{\nabla}_\alpha\gamma^\alpha\Psi - \frac{1}{2}\mu_0^2 \overset{\circ}{=} -\mu_0^2 \tag{7.33}$$

where the Dirac identity (7.29) has been used in the second step. This shows that the explicit contribution of the fundamental spinor field to the ground state energy vanishes. However, the first two terms on the right hand side of (7.32) also contribute, and the complete ground state energy density is obtained from the trace of (7.32):

$$(G^{cl})_\alpha{}^\alpha \overset{\circ}{=} -4(\Lambda^G + \Lambda^\Psi), \qquad \Lambda^\Psi = -\frac{1}{8}\mu_0^2 \tag{7.34}$$

In order to obtain a vanishing (or small) ground state energy on spacetime, the large negative contribution Λ^Ψ of the spinor field must be compensated by a corresponding positive contribution Λ^G of the gravitational field. This is not necessarily the result of a fine-tuning. Rather, the possibility is left open that the ground state expectation values are not constant on the whole Dirac manifold and spacetime is selected among other four-dimensional submanifolds by the condition that the ground state energy vanishes on $\mathcal{S}$ but takes Planck scale values on other submanifolds.

7.3 Constants of Nature

In order to compare the Heisenberg equation system in the vacuum approximation with the usual Einstein and Maxwell equations on spacetime, the fields, which up to now were dimensionless, must be equipped with their physical dimensions by multiplying them with suitable powers of a scale λ, the magnitude of which will be determined below:

$$\begin{aligned} &G_{\alpha\beta} \to \lambda^2 G_{\alpha\beta}. \qquad \mathcal{F}_{\alpha\beta} \to \lambda^2\mathcal{F}_{\alpha\beta}, \qquad T^{em}_{\alpha\beta} \to \lambda^4 T^{em}_{\alpha\beta} \\ &j^\alpha \to \lambda^3 j^\alpha, \qquad \nabla_\alpha \to \lambda\nabla_\alpha, \qquad \Psi \to \lambda^{\frac{3}{2}}\Psi, \qquad \mu_0 \to \lambda^{\frac{3}{2}}\mu_0 \end{aligned} \tag{7.35}$$

In terms of rescaled fields the gravitational Einstein equation and the Maxwell equations for F take the form:

$$
\begin{aligned}
&G_{\alpha\beta} \stackrel{\circ}{=} \frac{1}{2}\mu_0^{-1}\lambda^{\frac{1}{2}}\bar{\Psi}\overset{\leftrightarrow}{\nabla}_{(\alpha}\gamma_{\beta)}\Psi - \frac{1}{4}\mu_0^{-2}\lambda(\bar{\Psi}\gamma_5\gamma_\alpha\Psi)(\bar{\Psi}\gamma_5\gamma_\beta\Psi) + 8\mu_0^{-2}\lambda^{-1}T^{em}_{\alpha\beta} \\
&\nabla_\beta F^{\alpha\beta} \stackrel{\circ}{=} -\frac{1}{8}\mu_0\lambda^{\frac{3}{2}}j^\alpha, \qquad \nabla_\beta F^{*\alpha\beta} \stackrel{\circ}{=} 0
\end{aligned}
\tag{7.36}
$$

Apart from the quartic contribution to the energy-momentum tensor of the spinor field, these equations are equal to the Einstein and Maxwell equations for real electromagnetic and gravitational fields coupled to a Dirac spinor field, provided the constants may be expressed in terms of the Planck length λ_P and the fine structure constant α as follows:

$$
\mu_0^{-1}\lambda^{\frac{1}{2}} = g_0^2, \qquad 8\mu_0^{-2}\lambda^{-1} = g_0^2 g_1^{-2}, \qquad \frac{1}{8}\mu_0\lambda^{\frac{3}{2}} = g_1^2; \qquad g_0^2 = 8\pi\lambda_P^2, \qquad g_1^2 = 4\pi\alpha
\tag{7.37}
$$

These conditions yield three equations to determine λ and μ_0, which are however consistent and may be solved as follows:

$$
\mu_0^2 = 2\sqrt{2}g_1 g_0^{-3} = \frac{1}{4\pi}\sqrt{\alpha}\lambda_P^{-3}, \qquad \lambda = 2\sqrt{2}g_0 g_1 = 16\pi\sqrt{\alpha}\lambda_P
\tag{7.38}
$$

In terms of rescaled fields the electromagnetic Einstein equations and the Dirac identity take the form:

$$
\begin{aligned}
&F_{\alpha\beta} \stackrel{\circ}{=} \frac{i}{16}\lambda\,\bar{\Psi}[\gamma_\alpha, \gamma_\beta]\Psi + \frac{1}{4}\mu_0^{-1}\lambda^{\frac{1}{2}}\nabla_{[\alpha}j_{\beta]}, \qquad \widetilde{F}_{\alpha\beta} \stackrel{\circ}{=} \frac{1}{8}\mu_0^{-1}\lambda^{\frac{1}{2}}\epsilon_{\alpha\beta}{}^{\gamma\delta}\nabla_\gamma j_\delta \\
&\gamma^\alpha\nabla_\alpha\Psi + \frac{1}{8}\mu_0^{-1}\lambda^{\frac{1}{2}}(\bar{\Psi}\gamma_\alpha\Psi)\gamma^\alpha\Psi \stackrel{\circ}{=} 0
\end{aligned}
\tag{7.39}
$$

Inserting (7.38) into (7.36) and (7.39) one obtains the Maxwell equations (1.41), the electromagnetic (1.43) and gravitational (1.45) Einstein equations as well as the Dirac identity (1.46) in terms of α and λ_P.

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